УНИВЕРЗИТЕТ У НИШУ ФАКУЛТЕТ ЗАШТИТЕ НА РАДУ У НИШУ 18000 Ниш, Чарнојевића 10 А, Телефон: (018) 529-701, Факс: (018) 249-962, Т.Р. 840-1747666-77, ПИБ 100663853, М.Б. 07226063, www.znrfak.ni.ac.yu	**CURRICULUM VITAE НАСТАВНИКА - САРАДНИКА**

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1. M.S. Stanković, M.V. Vidanović, S.B. Tričković:
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4. M.S. Stanković, S.B. Tričković, M.V. Vidanović:
   Series over the product of Bessel and trigonometric functions,
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   Z. Anal. Anw. 20, No 1 (2001) 235-246. M23 - 3

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Appl. Anal. 80, No 1-2 (2001) 53-64. M23 - 3

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J. Comp. Anal. Appl. 5, No 3 (2003) 313-331. M23 - 3

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Series involving the product of a trigonometric integral and a trigonometric function,
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Радови саопштени на скупу међународног значаја штампани у целини :

12. M.S. Stanković, D.M. Petković, M.V. Đurić:
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13. G. Milovanović, M.S. Stanković, M.V. Vidanović:
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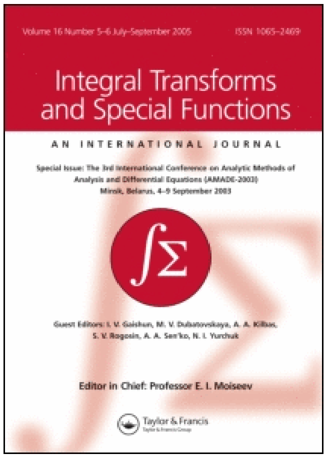

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Series involving the product of a trigonometric integral and a trigonometric function

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Series involving the product of a trigonometric integral and a trigonometric function

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This paper is concerned with the summation of series (1). To find the sum of the series (1) we first derive formulas for the summation of series whose general term contains a product of two trigonometric functions. These series are expressed in terms of Riemann's zeta, Catalan's beta function or Dirichlet functions eta and lambda, and in certain cases, thoroughly investigated here, they can be brought in closed form, meaning that the infinite series are represented by finite sums.

Keywords: Riemann's zeta; Catalan's beta function; Dirichlet; Bessel functions

Mathematics Subject Classification 1991: Primary: 33C10; Secondary: 11M06, 65B10

1. Introduction

In this paper we shall find the sum of the series

$$I_\alpha^{T,f} = \sum_{n=1}^{\infty} \frac{(s)^{n-1} T((an-b)x)}{(an-b)^\alpha} f((an-b)z), \quad \alpha \in \mathbb{R}^+, \tag{1}$$

where $a = \left\{ {1 \atop 2} \right\} b = \left\{ {0 \atop 1} \right\}$, $s = 1$ or -1, f is sin or cos, and T denotes trigonometric integral S_ϕ or C_ϕ defined by

$$S_\phi(x) = \int_0^1 \phi(y) \sin xy \, \mathrm{d}y, \quad C_\phi(x) = \int_0^1 \phi(y) \cos xy \, \mathrm{d}y. \tag{2}$$

To obtain the sum of the series (1) *we do not have to calculate integrals* $T((an-b)x)$ *previously*. At first, we assume that ϕ is integrable. Yet, in order to extend the class of

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Table 1.

a	b	s	c	F	for
1	0	1	1	ζ	$0 < x < 2\pi$
		−1	0	η	$-\pi < x < \pi$
2	1	1	1/2	λ	$0 < x < \pi$
		−1	0	β	$-(\pi/2) < x < (\pi/2)$

summable series, we admit that ϕ is differentiable on (0, 1), but not necessarily bounded in the neighborhood of 0 or 1, or not integrable on (0, 1), however, such that there exists at least one of the integrals (2). We further require that the functions $y^k\phi(y)$ $(k \in \mathbb{N})$ are integrable on (0, 1) as well.

Obtaining sums of the series (1) relies on the summation of some trigonometric series. Making use of the method for finding the formula (6) from our article [1] (see Theorem 1, p. 396), we can derive the other particular cases as well, writing all of them, in terms of Riemann's ζ or Catalan's β function or Dirichlet functions η and λ, in the form of a single formula, *i.e.*,

$$\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)x)}{(an-b)^{\alpha}} = \frac{c\pi}{2\Gamma(\alpha)f(\pi\alpha/2)}x^{\alpha-1} + \sum_{i=0}^{\infty}\frac{(-1)^i F(\alpha-2i-\delta)}{(2i+\delta)!}x^{2i+\delta}, \tag{3}$$

where $\alpha > 0$, $a = \left\{ {1 \atop 2} \right\}$ $b = \left\{ {0 \atop 1} \right\}$, $s = 1$ or -1, and $f = \left\{ {\sin \atop \cos} \right\}$ $\delta = \left\{ {1 \atop 0} \right\}$. The values for F and c are in the table 1, where ζ is Riemann's zeta function $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$, η and λ are Dirichlet functions $\eta(z) = \sum_{k=1}^{\infty}(-1)^{k-1}k^{-z} = (1-2^{1-z})\zeta(z)$, $\lambda(z) = \sum_{k=0}^{\infty}(2k+1)^{-z} = (1-2^{-z})\zeta(z)$, and β is Catalan's function $\beta(z) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-z}$.

Remark 1 We note that the functions ζ, η, λ are analytic in the whole complex plane except for $z = 1$, where they have a pole. The integral representation $\beta(z) = (1/\Gamma(z))\int_0^{\infty}(x^{z-1}e^x/(e^{2x}+1))\,dx$ of Catalan's function defines an analytical function for Re $z \geq 1$, but also it satisfies the functional equation $\beta(z) = (\pi/2)^{z-1}\Gamma(1-z)\cos(\pi z/2)\beta(1-z)$ extending beta to the left side of the complex plane Re $z < 1$.

Remark 2 After multiplying by $(2(x/2)^{\nu})/(\Gamma(1/2)\Gamma(\nu+(1/2)))$ the integrals (2) with $\phi(y) = (1-y^2)^{\nu-1/2}$, and substituting $y = \cos\theta$, we obtain

$$\varphi_{\nu}(z) = \frac{2(z/2)^{\nu}}{\Gamma(1/2)\Gamma(\nu+(1/2))}\int_0^{\pi/2}\sin^{2\nu}\theta\, g(z\cos\theta)\,d\theta, \tag{4}$$

where Re $\nu > -(1/2)$, $\varphi_{\nu} = \left\{ {J_{\nu} \atop \mathbf{H}_{\nu}} \right\}$ $g = \left\{ {\cos \atop \sin} \right\}$, J_{ν} and $\mathbf{H}_{\nu}$ are the Bessel and Struve functions respectively of the first kind and order ν. The integrals $S_{\phi}(x)$ and $C_{\phi}(x)$ in equation (2) can be considered as generalizations of Bessel and Struve functions, respectively, since their integral representations (4) are obtained when the particular function $\phi(y) = (1-y^2)^{\nu-1/2}$ is chosen. Just because of this, in this paper we generalize some of our results from [2].

2. Preliminaries

In order to find a sum of the series (1) we set one of the integrals (2) instead of $T(x)$, so that in the sequel we use denotations $T(x) = \left\{ {S_{\phi} \atop C_{\phi}} \right\}$ $g = \left\{ {\sin \atop \cos} \right\}$, meaning that g is one of the trigonometric functions in equation (2), which depends on the choice of S_ϕ or C_ϕ. Afterwards, we shall prove that we are allowed to interchange summation and integration, *i.e.*,

$$\begin{aligned} I_\alpha^{T,f} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z)}{(an-b)^\alpha} \int_0^1 \phi(y) g((an-b)xy)\,\mathrm{d}y \\ &= \int_0^1 \Big(\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z) g((an-b)xy)}{(an-b)^\alpha} \Big) \phi(y)\,\mathrm{d}y \quad (\alpha \in \mathbb{R}^+), \end{aligned} \tag{5}$$

relying on uniform convergence of the right-hand series with respect to y by virtue of Dirichlet's test, which says that (see [3]) the series $\sum_{n=0}^{\infty} a_n(y) b_n(y)$ is uniformly convergent in D, if the partial sums of $\sum_{n=0}^{\infty} a_n(y)$ are uniformly bounded in D, and the sequence $b_n(y)$, being monotonic for every fixed y, uniformly converges to 0. We emphasize that we treat the above right-hand series as a function of $y \in (0, 1)$, regarding x and z as variable parameters.

LEMMA 1 *Let $\phi(y)$ be integrable. Then the right-hand series converges uniformly and there holds equation* (5).

Proof We make use of an elementary trigonometric identity, whereby representing the product of f and g, as the sum of two trigonometric functions sin or cos. Now this series is split up into two series of the type

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} \tau((an-b)(z \pm xy))}{(an-b)^\alpha} \quad (\alpha \in \mathbb{R}^+), \tag{6}$$

where $\tau = \sin$ or $\tau = \cos$. Let us suppose first that $a = 1, b = 0$. If $s = 1$, then there holds

$$\left| \sum_{k=1}^{n} \tau(k(z \pm xy)) \right| \le \frac{1}{\sin((z \pm xy)/2)} \le \frac{1}{\sin \varepsilon}$$

for $0 < z \pm xy < 2\pi$, because $\sin((z \pm xy)/2) \ge \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \le ((z \pm xy)/2) \le \pi - \varepsilon$.

Similarly, if $s = -1$, we would have

$$\left| \sum_{k=1}^{n} (-1)^{k-1} \tau(k(z \pm xy)) \right| \le \frac{1}{\cos((z \pm xy)/2)} \le \frac{1}{\sin \varepsilon}$$

for $-\pi < z \pm xy < \pi$, because $\cos((z \pm xy)/2) \ge \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-(\pi/2) + \varepsilon \le ((z \pm xy)/2) \le (\pi/2) - \varepsilon$.

We now suppose $a = 2, b = 1$. If $s = 1$, then

$$\left| \sum_{k=1}^{n} \tau((2k-1)(z \pm xy)) \right| \le \frac{1}{\sin(z \pm xy)} \le \frac{1}{\sin \varepsilon}$$

for $0 < z \pm xy < \pi$, because $\sin(z \pm xy) \ge \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \le z \pm xy \le \pi - \varepsilon$. Finally, if $s = -1$, then

$$\left| \sum_{k=1}^{n} (-1)^{k-1} \tau((2k-1)(z \pm xy)) \right| \le \frac{1}{\cos(z \pm xy)} \le \frac{1}{\sin \varepsilon}$$

for $-(\pi/2) < z \pm xy < (\pi/2)$, because $\cos(z \pm xy) \geq \sin\varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-(\pi/2) + \varepsilon \leq z \pm xy \leq (\pi/2) - \varepsilon$.

So in all these cases, partial sums are uniformly bounded with respect to $y \in (0, 1)$, and on this basis, we determine values of x and z giving rise to this, thus finding boundaries for convergence regions of the series (1) in each of four considered cases. They are in table 2. First, we have immediately $-|x| \leq \pm xy \leq |x|$, and to obtain the condition $0 < z \pm xy < 2\pi$, it is necessary to take $|x| < z < 2\pi - |x|$, from where there follows $|x| < \pi$. In the second case, to satisfy $-\pi < z \pm xy < \pi$, it is necessary to take $|x| - \pi < z < \pi - |x|$, and we have again $|x| < \pi$. In the third, $0 < z \pm xy < \pi$, it is necessary to take $|x| < z < \pi - |x|$, so that we easily find $|x| < (\pi/2)$. Finally, for $-(\pi/2) < z \pm xy < (\pi/2)$, it is necessary to take $|x| - (\pi/2) < z < (\pi/2) - |x|$ giving again $|x| < (\pi/2)$.

On the other hand, the sequence $1/((an - b)^{\alpha})$ obviously monotonically tends to 0 for $\alpha > 0$, and uniformly converges to 0 with respect to y. Together with the above, this proves an uniform convergence of the right-hand series in equation (5) with respect to $y \in (0, 1)$. So the interchange of integration and summation in equation (5) is permitted, and accordingly, there holds equation (5). ■

LEMMA 2 *Suppose that, for a differentiable on* $(0, 1)$ *and unbounded in the neighborhood of* 0 *or* 1 *function* ϕ, *the integral* $\int_0^1 \phi(y)\mathrm{d}y$ *does not converge. Let there exist at least one of the integrals* $T((an - b)x)$ (T *is* S_ϕ *or* C_ϕ *defined by equation* (2)), *so that* $|T((an - b)x)| \leq M_n(x)$, *and for each corresponding x from table* 2, *the sequence* $(M_n(x))/((an - b)^{\alpha})$, $\alpha > 0$, *monotonically tends to* 0. *Then there holds* (5).

Proof Because of the assumption that ϕ is differentiable function, we know that it is continuous on each closed interval within (0, 1), and, as it is not bounded in the neighborhood of 0 or 1, without loss of generality, we can consider $[\delta, 1 - \delta]$, $0 < \delta < 1$. Continuous function on a closed interval is bounded, and referring again to Dirichlet's test, we prove uniform convergence of the right-hand series in equation (5) with respect to y, but this time on $[\delta, 1 - \delta]$, so that there holds

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z)}{(an-b)^{\alpha}} \int_{\delta}^{1-\delta} \phi(y) g((an-b)xy)\,\mathrm{d}y$$
$$= \int_{\delta}^{1-\delta} \left(\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z) g((an-b)xy)}{(an-b)^{\alpha}} \right) \phi(y)\,\mathrm{d}y \quad (\alpha \in \mathbb{R}^+). \tag{7}$$

We regard the left-hand series as a function of δ with variable parameters z and x. In view of the conditions, by virtue of Dirichlet's test, the left-hand series in equation (7) converges

Table 2.

a	b	s	c	F	Convergence regions
1	0	1	1	ζ	$\{(x, z) : -\pi < x < \pi,\ \|x\| < z < 2\pi - \|x\|\}$
1	0	-1	0	η	$\{(x, z) : -\pi < x < \pi,\ \|x\| - \pi < z < \pi - \|x\|\}$
2	1	1	1/2	λ	$\{(x, z) : -(\pi/2) < x < (\pi/2),\ \|x\| < z < \pi - \|x\|\}$
2	1	-1	0	β	$\{(x, z) : -(\pi/2) < x < (\pi/2),\ \|x\| - (\pi/2) < z < (\pi/2) - \|x\|\}$

uniformly with respect to δ on (0, 1). Hence, we have

$$\lim_{\delta\to 0+}\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)z)}{(an-b)^{\alpha}}\int_{\delta}^{1-\delta}\phi(y)g((an-b)xy)\,\mathrm{d}y$$
$$=\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)z)}{(an-b)^{\alpha}}\int_{0}^{1}\phi(y)g((an-b)xy)\,\mathrm{d}y,$$

meaning that the right-hand integral in equation (7) converges, so there holds equation (5). ∎

3. Series over the product of trigonometric functions

We have seen that in finding the summation formula for equation (1), the key role plays the series including the product of two trigonometric functions. That is why we are going to investigate them thoroughly. Now, by applying equation (3) to both series in each of the above particular cases, we obtain the following general formula

$$S_{\alpha}^{f,g}=\sum_{n=1}^{\infty}\frac{(s)^{n-1}g((an-b)xy)\,f((an-b)z)}{(an-b)^{\alpha}}$$
$$=\frac{c\pi(-1)^{\delta(\delta-d)}}{4\Gamma(\alpha)h(\pi\alpha/2)}\left((z+xy)^{\alpha-1}+(-1)^{\delta}(z-xy)^{\alpha-1}\right)$$
$$+\sum_{i=0}^{\infty}\frac{(-1)^{\delta(\delta-d)+i}F(\alpha-2i-d)}{(2i+d)!}\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}(xy)^{2j+\delta}, \quad (8)$$

where $g=\left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$, and $d=\left\{\begin{smallmatrix}0 & f=g\\ 1 & f\neq g\end{smallmatrix}\right.$, and also there holds $h=\left\{\begin{smallmatrix}\cos & f=g\\ \sin & f\neq g\end{smallmatrix}\right.$. All the other relevant parameters are in table 2.

3.1 *Limiting values of equation (8)*

When on the right-hand side of equation (8) appears $h=\sin$ and $\alpha=2m$ or $h=\cos$ and $\alpha=2m-1$, where $m\in\mathbb{N}$, one should take limit. Let us consider a particular case of the formula (8), taking $a=1$, $b=0$, $s=1$, which implies $c=1$, $F=\zeta$. If $g=\cos$ and $f=\sin$ then $\delta=0$, $h=\sin$, $d=1$, so we have

$$\sum_{n=1}^{\infty}\frac{\sin nz\cos nxy}{n^{\alpha}}=\pi\frac{(z+xy)^{\alpha-1}+(z-xy)^{\alpha-1}}{4\Gamma(\alpha)\sin(\pi\alpha/2)}$$
$$+\sum_{i=0}^{\infty}\frac{(-1)^{i}\zeta(\alpha-2i-1)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j}z^{2i-2j+1}(xy)^{2j}.$$

As we have said, we take limit

$$\lim_{\alpha\to 2m}\left[\pi\frac{(z+xy)^{\alpha-1}+(z-xy)^{\alpha-1}}{4\Gamma(\alpha)\sin(\pi\alpha/2)}\right.$$
$$\left.+\sum_{i=0}^{m-1}\frac{(-1)^{i}\zeta(\alpha-2i-1)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j}z^{2i-2j+1}(xy)^{2j}\right]=G_{2m}(x,y,z),$$

where we have found

$$G_{2m}(x,y,z) = \frac{(-1)^m}{2(2m-1)!}\big((z+xy)^{2m-1}(\log(z+xy)-\psi(2m)-\gamma)$$
$$+ (z-xy)^{2m-1}(\log(z-xy)-\psi(2m)-\gamma)\big)$$
$$+\sum_{k=1}^{m-1}\frac{(-1)^{k-1}\zeta(2m-2k+1)}{(2k-1)!}\sum_{j=0}^{k-1}\binom{2k-1}{2j}z^{2k-2j-1}(xy)^{2j},$$

where γ is Euler's constant and ψ is the digamma function, $\psi(s) = (\Gamma'(s)/\Gamma(s))$, whose relation to the harmonic numbers $H_n = \sum_{j=1}^{n}(1/j)$ is $\psi(n) = H_{n-1} - \gamma$, with $\psi(1) = -\gamma = \Gamma'(1)$. Finally,

$$\sum_{n=1}^{\infty}\frac{\sin nz\cos nxy}{n^{2m}} = G_{2m}(x,y,z) + \sum_{i=m}^{\infty}\frac{(-1)^i\zeta(2m-2i-1)}{(2i+1)!}$$
$$\times\sum_{j=0}^{i}\binom{2i+1}{2j}z^{2i-2j+1}(xy)^{2j}.$$

Example 1 If we additionally choose $m = 1$, we obtain

$$\sum_{n=1}^{\infty}\frac{\sin nz\cos nxy}{n^{2}} = z - \frac{1}{2}\left(z\log(z^2-(xy)^2) + xy\log\frac{z+xy}{z-xy}\right)$$
$$+\sum_{i=1}^{\infty}\frac{(-1)^i\zeta(1-2i)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j}z^{2i-2j+1}(xy)^{2j}.$$

In the same way, we can come to a similar result for any of particular cases if $h = \sin$ and $\alpha = 2m$. Also, we repeat the same procedure, if $h = \cos$ and $\alpha = 2m-1$.

Example 2 If we choose the same parameters as above, but take $f = \cos$ instead of $f = \sin$ and again $m = 1$, then $h = \cos$ and $\alpha = 1$, so we first have to find the limiting value

$$\lim_{\alpha\to 1}\left(\pi\frac{(z+xy)^{\alpha-1}+(z-xy)^{\alpha-1}}{4\Gamma(\alpha)\sin(\pi\alpha/2)} + z\zeta(\alpha)\right) = -\frac{1}{2}\log(z^2-(xy)^2),$$

so that we have

$$\sum_{n=1}^{\infty}\frac{\cos nxy\cos nz}{n} = -\frac{1}{2}\log(z^2-(xy)^2) + \sum_{i=1}^{\infty}\frac{(-1)^i\zeta(1-2i)}{(2i)!}\sum_{j=0}^{i}\binom{2i}{2j}z^{2i-2j}(xy)^{2j}.$$

However, based on the Fourier expansion of the function $-(1/2)\log(2\sin(t/2))$ for $0 < t < 2\pi$, there holds $-(1/2)\log(2\sin(t/2)) = \sum_{n=1}^{\infty}(1/n)\cos nt$, and we can easily find

$$\sum_{n=1}^{\infty}\frac{\cos nxy\cos nz}{n} = -\frac{1}{2}\log\left(4\sin\frac{z-xy}{2}\sin\frac{z+xy}{2}\right),$$

where $0 < z \pm xy < 2\pi$. Relying on this result, we can conclude

$$\sum_{i=1}^{\infty}\frac{(-1)^i\zeta(1-2i)}{(2i)!}\sum_{j=0}^{i}\binom{2i}{2j}z^{2i-2j}(xy)^{2j} = \log\sqrt{\frac{(z-xy)(z+xy)}{4\sin((z-xy)/2)\sin((z+xy)/2)}},$$

which is a new summation formula for the series involving Riemann's zeta function. Further, if we let $xy \to z$, and make use of the combinatorial identity $\sum_{j=0}^{i}\binom{2i}{2j} = 2^{2i-1}$, the above summation formula will be reduced to

$$\sum_{i=1}^{\infty} \frac{(-4z^2)^i \zeta(1-2i)}{(2i)!} = \log\left(\frac{z}{\sin z}\right).$$

3.2 *Closed form cases and some applications of equation* (8)

If $\alpha - d = 2m$ and $F = \zeta, \eta, \lambda$ or $\alpha - d = 2m - 1$ and $F = \beta$ $(m \in \mathbb{N})$, the sum of the series on the right-hand side of equation (8) consists of a finite number of terms because the functions ζ, η, λ vanish at even negative integers, and the function β vanishes at odd negative integers. We shall denote $\alpha = 2m + d - \varepsilon$, where $\varepsilon = \begin{cases} 0, & F=\zeta,\eta,\lambda \\ 1, & F=\beta \end{cases}$. So, for this choice of parameters, the formula (8) is brought into a so-called closed form

$$\begin{aligned} S^{f,g}_{2m+d-\varepsilon} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} g((an-b)xy)\, f((an-b)z)}{(an-b)^{2m+d-\varepsilon}} \\ &= \frac{c\pi}{2} \sum_{j=0}^{m} \binom{2m+d-\varepsilon-1}{2j+\delta} \frac{(-1)^{\delta(\delta-d)} z^{2m-2j+d-\delta-\varepsilon-1} (xy)^{2j+\delta}}{(2m+d-\varepsilon-1)!\, h(m\pi + ((d-\varepsilon)\pi)/2)} \\ &\quad + \sum_{i=0}^{m} \frac{(-1)^{\delta(\delta-d)+i} F(2m-2i-\varepsilon)}{(2i+d)!} \sum_{j=0}^{i} \binom{2i+d}{2j+\delta} z^{2i-2j+d-\delta} (xy)^{2j+\delta}, \end{aligned} \tag{9}$$

where $g = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$ $\delta = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}$, and independently of that $d = \begin{cases} 0 & f=g \\ 1 & f\neq g \end{cases}$, $h = \begin{cases} \cos & f=g \\ \sin & f\neq g \end{cases}$. The other relevant parameters are in table 2. The formula (9) comprises some particular results from [4], but it is more suitable for immediately obtaining sums of some infinite series as well. For example, in [2], §5.4.15, entry 8, we find

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin nxy \cos nz = \frac{xy}{12}\left(\pi^2 - (xy)^2 - 3z^2\right).$$

It holds if $|xy \pm z| \le \pi$, which is the condition for uniform convergence of the left-hand side series in equation (8) with $a = 1, b = 0$ and $s = -1$. After referring to table 2 and the conditions following equation (9), we obtain the above formula, placing in equation (9): $g = \sin,\ f = \cos,\ m = 1, d = 1, \varepsilon = 0, c = 0,\ F = \eta, \delta = 1, h = \sin$.

3.3 *Some applications*

In [5] the solution of the boundary value problem

$$U''_{tt} = a^2 U''_{xx}, \quad U(x,0) = \frac{4hx(L-x)}{L^2}, \quad U'_t(x,0) = 0, \ \ 0 \le x \le L, \ \ t \ge 0,$$

is given by the infinite series

$$U(x,t) = \frac{32h}{\pi^3} \sum_{n=1}^{\infty} \frac{\cos((\pi(2n-1)at)/L) \sin(((2n-1)\pi x)/L)}{(2n-1)^3}.$$

However, using equation (9), where we take $m = 1,\ a = 2,\ b = 1,\ s = 1,\ c = (1/2)$ and $F = \lambda$, then $\delta = 0,\ d = 1$ and $\varepsilon = 0$, and denoting $(at\pi/L) = xy$, $(x\pi/L) = z$, we obtain its solution in closed form

$$U(x,t) = \frac{4h}{L^2}(xL - x^2 - a^2t^2),\ \ 0 \le \frac{at}{L} \le \frac{1}{2},\ \ \left|\frac{at}{L}\right| \le \frac{x}{L} \le 1 - \left|\frac{at}{L}\right|.$$

Also (see [5]), the solution of the boundary value problem

$$\begin{aligned} U''_{tt} &= U''_{xx} + x(x-L), \quad U(x,0) = U'_t(x,0) \\ &= U(0,t) = U(L,t) = 0,\ \ 0 \le x \le L,\ t \ge 0, \end{aligned}$$

is

$$U(x,t) = \frac{8L^4}{\pi^5}\sum_{n=1}^{\infty}\frac{\cos(\pi t(2n-1)/L)\sin(\pi x(2n-1)/L)}{(2n-1)^5} - \frac{x}{12}(x^3 - 2x^2L + L^3).$$

Applying equation (9) with $m = 2$, using the same parameters as above, but denoting $(t\pi/L) = xy$ and $(x\pi/L) = z$, this solution in closed form becomes

$$U(x,t) = \frac{1}{2}x^2t^2 - \frac{1}{2}Lxt^2 + \frac{1}{12}t^4, \quad 0 \le t \le \frac{L}{2},\ \ t \le x \le L - t.$$

4. Sum of the series (1)

Now we make use of equation (8), which is, as we have shown, the formula for finding sum of the right-hand side series of equation (5). We develop in equation (8) the binomials $(z \pm xy)^{\alpha-1}$ into binomial series, and after a rearrangement we actually obtain the summation formula of the left-hand side series in equation (5), which is actually the summation formula for the series (1)

$$\begin{aligned} I_{\alpha}^{T,f} &= \frac{c\pi(-1)^{\delta(\delta-d)}}{4\Gamma(\alpha)h(\pi\alpha/2)}\sum_{k=1}^{\infty}\binom{\alpha-1}{2k-\delta} z^{\alpha-1-2k+\delta}x^{2k-\delta}\int_0^1 y^{2k-\delta}\phi(y)\mathrm{d}y \\ &\quad + \sum_{i=0}^{\infty}\frac{(-1)^{\delta(\delta-d)+i}F(\alpha-2i-d)}{(2i+d)!}\sum_{j=0}^{i}\binom{2i+d}{2j+\delta} z^{2i-2j+d-\delta}x^{2j+\delta}\int_0^1 y^{2j+\delta}\phi(y)\mathrm{d}y, \end{aligned} \tag{10}$$

where $T(x) = \left\{ \begin{smallmatrix} S_\phi(x) \\ C_\phi(x) \end{smallmatrix} \right\}$ $g = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$ $\delta = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}$, $h = \begin{cases} \cos, & f=g \\ \sin, & f \ne g \end{cases}$ and $d = \begin{cases} 0, & f=g \\ 1, & f \ne g \end{cases}$. The other relevant parameters are in table 2.

4.1 *Limiting values*

If $h = \sin$ and $\alpha = 2m$ or $h = \cos$ and $\alpha = 2m - 1$ ($m \in \mathbb{N}$), we encounter a singularity in the first term of equation (10). That is why the limit should be taken. We shall demonstrate it for the following choice of parameters: $a = 1,\ b = 0,\ s = 1,\ c = 1,\ F = \zeta,\ g = \sin,\ f = \cos,\ \delta = 1,\ h = \sin$ and $d = 1$, and, apart from this, $\phi(y) = y^{-1}$, so that equation (10) becomes

$$I_\alpha^{S_\phi,\cos} = \sum_{n=1}^{\infty} \frac{\cos nz}{n^\alpha} \int_0^1 \frac{\sin nxy}{y}\,\mathrm{d}y = \frac{\pi}{2\Gamma(\alpha)\sin(\pi\alpha/2)} \sum_{k=1}^{\infty} \binom{\alpha-1}{2k-1} \frac{z^{\alpha-2k}x^{2k-1}}{2k-1}$$
$$+ \sum_{i=0}^{\infty} \frac{(-1)^i \zeta(\alpha-2i-1)}{(2i+1)!} \sum_{j=0}^{i} \binom{2i+1}{2j+1} \frac{z^{2i-2j}x^{2j+1}}{2j+1}.$$

Having chosen $\phi(y) = 1/y$, the integral $\int_0^1 (\mathrm{d}y/y)$ does not converge, but there exists the integral $\int_0^1 (\sin nxy/y)\mathrm{d}y = \mathrm{Si}(nx)$, where $\mathrm{Si}(z) = \int_0^z (\sin t/t)\mathrm{d}t$ is the integral sine, which could not be elementarily calculated, but we know it is bounded by $\pi/2$, so Lemma 2 holds. Irrespective of this, without even trying to find its value, and simply by applying equation (10), we obtain the above sum. Yet, because of $h = \sin$, there still remains to take the limit for $\alpha = 2m,\ m \in \mathbb{N}$, *i.e.*,

$$\lim_{\alpha\to 2m} \left[\frac{\pi}{2\Gamma(\alpha)\sin(\pi\alpha/2)} \sum_{k=1}^{m} \binom{\alpha-1}{2k-1} \frac{z^{\alpha-2k}x^{2k-1}}{2k-1} \right.$$
$$\left. + \sum_{i=0}^{m-1} \frac{(-1)^i \zeta(\alpha-2i-1)}{(2i+1)!} \sum_{j=0}^{i} \binom{2i+1}{2j+1} \frac{z^{2i-2j}x^{2j+1}}{2j+1} \right] = G_{2m}(x,z),$$

and we find

$$G_{2m}(x,z) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}\zeta(2m-2k+1)}{(2k-1)!} \sum_{j=0}^{k-1} \binom{2k-1}{2j+1} \frac{z^{2k-2j-2}x^{2j+1}}{2j+1} + \frac{(-1)^m}{(2m-1)!}$$
$$\times \left(\sum_{k=1}^{m-1} \frac{\log z - \psi(2m-2k+1) - \gamma}{2k-1} \binom{2m-1}{2k-1} x^{2k-1} z^{2m-2k} + \frac{x^{2m-1}\log z}{2m-1} \right)$$

as well as

$$\lim_{\alpha\to 2m} \frac{\pi}{2\Gamma(\alpha)\sin(\pi\alpha/2)} \sum_{k=m+1}^{\infty} \binom{\alpha-1}{2k-1} \frac{z^{\alpha-2k}x^{2k-1}}{2k-1}$$
$$= \frac{(-1)^{m-1}z^{2m-1}}{(2m)!} \sum_{k=m+1}^{\infty} \frac{(x/z)^{2k-1}}{\binom{2k-1}{2m}(2k-1)},$$

so that we get

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^{2m}} \int_0^1 \frac{\sin nxy}{y}\,\mathrm{d}y = G_{2m}(x,z) + \frac{(-1)^{m-1}z^{2m-1}}{(2m)!} \sum_{k=m+1}^{\infty} \frac{(x/z)^{2k-1}}{\binom{2k-1}{2m}(2k-1)}$$
$$+ \sum_{i=m}^{\infty} \frac{(-1)^i \zeta(2m-2i-1)}{(2i+1)!} \sum_{j=0}^{i} \binom{2i+1}{2j+1} \frac{z^{2i-2j}x^{2j+1}}{2j+1}.$$

For instance,

$$\sum_{n=1}^{\infty}\frac{\cos nz}{n^4}\int_0^1\frac{\sin nxy}{y}\,\mathrm{d}y=-\frac{11x^3}{108}-\frac{19\,x\,z^2}{18}+\left(\frac{x^2\,z}{4}+\frac{11z^3}{36}\right)\mathrm{arcth}\frac{x}{z}$$
$$+\frac{x^5}{24z^2}\Phi\left(\frac{x^2}{z^2},2,\frac{5}{2}\right)+\left(\frac{x^3}{36}+\frac{x\,z^2}{4}\right)\log\left(1-\frac{x^2}{z^2}\right)$$
$$+\left(\frac{x^3}{18}+\frac{x\,z^2}{2}\right)\log z+x\,\zeta(3)$$
$$+\sum_{i=2}^{\infty}\frac{(-1)^i\zeta(3-2i)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j+1}\frac{z^{2i-2j}x^{2j+1}}{2j+1},$$

where Φ is the Lerch transcendent function defined by the series $\Phi(z,s,\alpha)=\sum_{n=0}^{\infty}(z^n/(n+\alpha)^s)$, and $|x|<\pi$, $|x|<z<2\pi-|x|$ (see table 2).

In the same way, we can come to a similar result for any of particular cases.

4.2 *Closed form cases*

They ensue if $\alpha-d=2m$ and $F=\zeta,\eta,\lambda$ or $\alpha-d=2m-1$ and $F=\beta$ ($m\in\mathbb{N}$). The other relevant parameters and convergence regions are in table 2. When we choose the function ϕ in equation (10), we additionally have to calculate both integrals having in fact a similar structure.

We have already seen that the formula (9) contains all closed form cases for the product of trigonometric functions. So, using the formula (9) we bring the sum of the series (1) into closed form:

$$I_{2m+d-\varepsilon}^{T,f}=\sum_{n=1}^{\infty}\frac{(s)^{n-1}T((an-b)x)}{(an-b)^{2m+d-\varepsilon}}f((an-b)z)$$
$$=\frac{c\pi}{2}\sum_{j=0}^{m}\binom{2m+d-\varepsilon-1}{2j+\delta}\frac{(-1)^{\delta(\delta-d)}z^{2m-2j+d-\delta-\varepsilon-1}x^{2j+\delta}}{(2m+d-\varepsilon-1)!\,h(m\pi+((d-\varepsilon)\pi)/2)}$$
$$\times\int_0^1 y^{2j+\delta}\phi(y)\mathrm{d}y+\sum_{i=0}^{m}\frac{(-1)^{\delta(\delta-d)+i}F(2m-2i-\varepsilon)}{(2i+d)!}$$
$$\times\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}x^{2j+\delta}\int_0^1 y^{2j+\delta}\phi(y)\mathrm{d}y, \tag{11}$$

where $T(x)=\left\{\begin{smallmatrix}S_\phi(x)\\C_\phi(x)\end{smallmatrix}\right\}$ $g=\left\{\begin{smallmatrix}\sin\\\cos\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\}$, and independently of that $d=\begin{cases}0 & f=g\\1 & f\neq g\end{cases}$, $h=\begin{cases}\cos & f=g\\\sin & f\neq g\end{cases}$. For $F=\zeta,\eta,\lambda$ there holds $\varepsilon=0$, but for $F=\beta$ it is $\varepsilon=1$. The other relevant parameters are in table 2.

Example 3 First we take $a=1$, $b=0$ and $s=1$ in equation (11), there follows $c=1$ and $F=\zeta$ (see table 2), then $\varepsilon=0$. We choose $\phi(y)=\mathrm{ctg}y$. For $g=\cos$ the integral $\int_0^1\mathrm{ctg}y\cos nxy\,\mathrm{d}y$ does not exist, so there must be $g=\sin$, which means $\delta=1$. If we take

$f = \cos$, then $h = \sin$ and $d = 1$ because $f \neq g$. Let $m = 1$. For $0 < |x| < \pi$, we have

$$\left| x \int_0^1 \operatorname{ctg} y \sin nxy \, \mathrm{d}y \right| = \left| \int_0^x \operatorname{ctg}(t/x) \sin nt \, \mathrm{d}t \right| \leqslant \int_0^x \left| \operatorname{ctg}(t/x) \sin nt \right| \mathrm{d}t$$
$$\leqslant \int_0^\pi \left| \operatorname{ctg}(t/\pi) \sin nt \right| \mathrm{d}t.$$

Also, $|\operatorname{tg}(t/\pi)| > (|t|/\pi)$ implies $|\operatorname{ctg}(t/\pi)| < (\pi/|t|)$, for $|t| < \pi$. Additionally, $\lim_{t\to 0+} \operatorname{ctg}(t/\pi) \sin nt = n\pi$. The function $|\operatorname{ctg}(t/\pi)\sin nt|$ is non-negative, fast oscillatory, and its maximal value between two consecutive zeros constantly decreases. There follows

$$\int_0^\pi \left| \operatorname{ctg}\frac{t}{\pi} \sin nt \right| \mathrm{d}t = \int_0^{1/n} \left| \operatorname{ctg}\frac{t}{\pi} \sin nt \right| \mathrm{d}t + \int_{1/n}^\pi \left| \operatorname{ctg}\frac{t}{\pi} \right| \cdot \left| \sin nt \right| \mathrm{d}t$$
$$< \int_0^{1/n} n\pi \, \mathrm{d}t + \pi \int_{1/n}^\pi \frac{\mathrm{d}t}{t} = \pi(1 + \log n\pi),$$

which means that $\left| \int_0^1 \operatorname{ctg} y \sin nxy \, \mathrm{d}y \right| < (\pi(1+\log n\pi)/|x|) = M_n(x)$, and we can easily see that for $\alpha > 0$ and each x, $0 < |x| < \pi$, the sequence $(M_n(x)/n^\alpha) \to 0$ monotonically, so that we may apply Lemma 2. Also, we find

$$\int_0^1 y^{2j+1} \operatorname{ctg} y \, \mathrm{d}y = \begin{cases} \log(2\sin 1) + (1/2)\mathrm{Cl}_2(2), & j = 0 \\ \log(2\sin 1) + (3/2)\mathrm{Cl}_2(2) + (3/2)\mathrm{Cl}_3(2) - (3/4)\mathrm{Cl}_4(2), & j = 1 \end{cases}$$

where on the right-hand side are Clausen functions defined by (see [1])

$$\mathrm{Cl}_{2\nu}(x) = \sum_{n=1}^\infty \frac{\sin nx}{n^{2\nu}}, \quad \mathrm{Cl}_{2\nu-1}(x) = \sum_{n=1}^\infty \frac{\cos nx}{n^{2\nu-1}}, \quad \nu \in \mathbb{N}.$$

Making use of equation (9), we obtain

$$\begin{aligned} I_1^{S_\phi,\cos} &= \sum_{n=1}^\infty \frac{\cos nz}{n^3} \int_0^1 \operatorname{ctg} y \sin nxy \, \mathrm{d}y \\ &= \frac{x}{12}(2\pi^2 - 6\pi z + 3z^2)\left(\log(2\sin 1) + \frac{1}{2}\mathrm{Cl}_2(2)\right) \\ &\quad + \frac{x^3}{12}\left(\log(2\sin 1) + \frac{3}{2}\mathrm{Cl}_2(2) + \frac{3}{2}\mathrm{Cl}_3(2) - \frac{3}{4}\mathrm{Cl}_4(2)\right), \end{aligned}$$

where $|x| < \pi$ and $|x| < z < 2\pi - |x|$ (see table 2).

Example 4 Let $a = 1, b = 0$ and $s = -1$ in equation (11), implying $c = 0$, $F = \eta$ and $\varepsilon = 0$. For $g = \cos$ there must be $\delta = 0$. Further, we take $f = \cos$, there follows $h = \cos$, $d = 0$ because $f = g$. Let $m = 2$. We choose $\phi(y) = (1 - y^2)^{-1/2}$. It is unbounded about 1, but integrable on (0, 1), thus satisfying conditions of Lemma 1. So applying equation (11), we find

$$\begin{aligned} I_4^{C_\phi,\cos} &= \sum_{n=1}^\infty (-1)^{n-1} \frac{\cos nz}{n^4} \int_0^1 \frac{\cos nxy}{\sqrt{1-y^2}} \, \mathrm{d}y \\ &= \frac{7\pi^5}{1440} - \frac{\pi^3}{96}(x^2 + 2z^2) + \frac{\pi}{32}\left(\frac{x^4}{8} + x^2 z^2 + \frac{z^4}{3}\right), \end{aligned}$$

where $|x| < \pi$ and $|x| - \pi < z < \pi - |x|$ (see table 2).

Remark 3 We note that this particular result can be obtained in a different way. Namely, substituting $y = \cos\theta$, then multiplying with $2/\pi$, we deal with Bessel functions (see equation (4)), and the above series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} J_0(nx)}{n^4} \cos nz,$$

which is the series over the product of Bessel and trigonometric function, and its sum can be obtained by means of the formula on the page 289 in [2].

Example 5 Further, we take $a = 2$, $b = 1$ and $s = 1$ in equation (11). In table 2, we read $c = (1/2)$ and $F = \lambda$. So $\varepsilon = 0$. If we choose $g = \cos$, $f = \sin$, then we have $\delta = 0$, $d = 1$ and $h = \sin$. Let $m = 1$ and $\phi(y) = \log(\sin y)$. It is unbounded in the neighborhood of 0, however $\int_0^1 \log(\sin y)\,\mathrm{d}y = -\log 2 - (1/2)\mathrm{Cl}_2(2)$, and Lemma 1 holds. Applying equation (11), we obtain

$$\begin{aligned} I_1^{S_\phi,\sin} &= \sum_{n=1}^{\infty} \frac{\sin((2n-1)z)}{(2n-1)^3} \int_0^1 \log(\sin y)\cos((2n-1)xy)\,\mathrm{d}y \\ &= \frac{\pi}{96}\Big(6z(z-\pi)(\mathrm{Cl}_2(2)+\log 4) + x^2(6\mathrm{Cl}_2(2)+6\mathrm{Cl}_3(2)-3\mathrm{Cl}_4(2)+\log 16)\Big), \end{aligned}$$

where $|x| < (\pi/2)$ and $|x| - (\pi/2) < z < (\pi/2) - |x|$ (see table 2).

Example 6 Finally, we take in equation (11): $a = 2$, $b = 1$, $s = -1$, $c = 0$, $F = \beta$, $g = \sin$, $f = \sin$, $d = 0$, $\delta = 1$, $h = \cos$, $\varepsilon = 1$ and $m = 2$, and choose $\phi(y) = \log y$. The integral $\int_0^1 \log y\,\mathrm{d}y$ converges, ϕ meets requirements of Lemma 1, and making use of equation (11), the infinite series is brought in closed form

$$\begin{aligned} I_5^{S_\phi,\sin} &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin((2n-1)z)}{(2n-1)^5} \int_0^1 \log y \sin((2n-1)xy)\,\mathrm{d}y \\ &= \frac{\pi x z}{32}\left(\frac{\pi^2}{4} - \frac{x^2}{12} - \frac{z^2}{3}\right), \end{aligned}$$

where $|x| < (\pi/2)$ and $|x| - (\pi/2) < z < (\pi/2) - |x|$ (see table 2).

This series can take another form. At first, by integrating by parts, we come to the relation $\int_0^1 \log y \sin((2n-1)xy)\,\mathrm{d}y = (1/((2n-1)x)) \int_0^1 ((\cos((2n-1)xy) - 1/y)\,\mathrm{d}y$. Afterwards, we prove $\int_0^p ((1-\mathrm{e}^{-t})/t)\mathrm{d}t - \int_p^\infty (\mathrm{e}^{-t}/t)\,\mathrm{d}t = \log p + \gamma$. On this basis, we find $\mathrm{Ci}(p) + \mathrm{Cin}(p) = \log p + \gamma$, where $\mathrm{Ci}(p) = -\int_p^\infty (\cos t/t)\mathrm{d}t$ is the integral cosine and $\mathrm{Cin}(p) = \int_0^p ((1-\cos t)/t)\mathrm{d}t$ its related function. Hence, there follows

$$\begin{aligned} &\int_0^1 \log y \sin((2n-1)xy)\,\mathrm{d}y \\ &\quad = \frac{1}{(2n-1)x}\big(\mathrm{Ci}((2n-1)x) - \gamma - \log((2n-1)x)\big) \quad (x > 0), \end{aligned}$$

and we obtain the sum of a new series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin((2n-1)z)}{(2n-1)^6 x}\big(\mathrm{Ci}((2n-1)x)-\gamma-\log((2n-1)x)\big)$$
$$=\frac{\pi xz}{32}\left(\frac{\pi^2}{4}-\frac{x^2}{12}-\frac{z^2}{3}\right).$$

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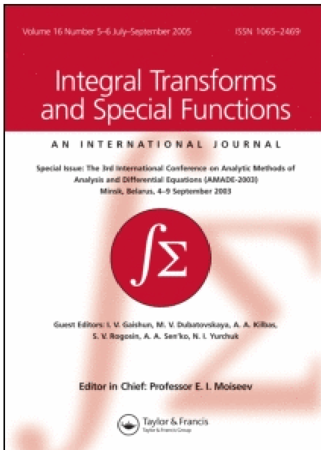

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On the summation of trigonometric series

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On the summation of trigonometric series

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We deal with the series

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^{\alpha}}, \quad \alpha \in \mathbb{R}^{+},$$

and express it as a power series in terms of Riemann's ζ or Catalan's β function or Dirichlet functions η and λ. Also, closed form cases as well as those when it is necessary to take limit have been thoroughly analyzed. Some applications such as convergence acceleration are considered too.

Keywords: Riemann's ζ and Catalan's β function; Dirichlet η and λ functions

2000 Mathematics Subject Classification: Primary: 33C10; Secondary: 11M06, 65B10

1. Introduction and preliminaries

There have been many particular cases (see, for example, [2] and [7]) of the trigonometric series of the type

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1}\, f((an-b)x)}{(an-b)^{\alpha}}, \quad \alpha \in \mathbb{R}^{+}, \tag{1}$$

where s is 1 or -1, $f = \sin$ or $f = \cos$ and $a = \{{}^{1}_{2}\} b = \{{}^{0}_{1}\}$. We first derive a general formula for finding the sum of Equation (1), meaning that we consider α to be a positive real parameter with the exclusion of positive integers. Afterwards we regard α as a positive integer, whereupon there appears a necessity to distinguish between the cases when infinite series (1) reduces to a finite sum, and those when limiting value must be taken. Apart from all known particular cases, the general formula yields new ones. We start with the following

LEMMA 1 *Series* (1) *is uniformly convergent for* $\alpha > 0$. *Convergence regions are given in Table* 1, *where* ζ *is Riemann's zeta function* $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$, η *and* λ *are Dirichlet functions* $\eta(z) = \sum_{k=1}^{\infty} (-1)^{k-1} k^{-z} = (1 - 2^{1-z})\zeta(z)$, $\lambda(z) = \sum_{k=0}^{\infty} (2k+1)^{-z} = (1 - 2^{-z})\zeta(z)$, *and* β *is Catalan's beta function* $\beta(z) = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-z}$.

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Table 1. Parameters and convergence region

a	b	s	c	F	f	δ	r	$\alpha \in$	Convergence region
1	0	1	1	ζ	sin	1	$2m-1$	$\mathbb{R}^+ \setminus 2\mathbb{N}$	$0 < x < 2\pi$
					cos	0	$2m$	$\mathbb{R}^+ \setminus 2\mathbb{N}-1$	
		-1	0	η	sin	1	$2m-1$	$\mathbb{R}^+ \setminus 2\mathbb{N}$	$-\pi < x < \pi$
					cos	0	$2m$	$\mathbb{R}^+ \setminus 2\mathbb{N}-1$	
2	1	1	1/2	λ	sin	1	$2m-1$	$\mathbb{R}^+ \setminus 2\mathbb{N}$	$0 < x < \pi$
					cos	0	$2m$	$\mathbb{R}^+ \setminus 2\mathbb{N}-1$	
		-1	0	β	sin	1	$2m$	$\mathbb{R}^+ \setminus 2\mathbb{N}-1$	$-\pi/2 < x < \pi/2$
					cos	0	$2m-1$	$\mathbb{R}^+ \setminus 2\mathbb{N}$	

Remark 1 We note that ζ, η, λ are analytic in the whole complex plane except for $z = 1$, where they have a pole, whereas Catalan's beta function $\beta(z) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-z}$ satisfies the functional equation $\beta(z) = (\pi/2)^{z-1}\Gamma(1-z)\cos(\pi z/2)\beta(1-z)$ extending the beta function to the left side of the complex plane $Re\, z < 1$.

Proof We shall make use of Dirichlet's test saying that (see [3]) the series $\sum_{n=0}^{\infty} a_n(x)b_n(x)$ is uniformly convergent in D, if the partial sums of $\sum_{n=0}^{\infty} a_n(x)$ are uniformly bounded in D, and the sequence $b_n(x)$, being monotonic for every fixed x, uniformly converges to 0.

Let us suppose first that $a = 1, b = 0$. If $s = 1$, then there holds

$$\left|\sum_{k=1}^{n} f(kx)\right| \le \frac{1}{\sin(x/2)} \le \frac{1}{\sin\varepsilon}$$

for $0 < x < 2\pi$, because $\sin(x/2) \ge \sin\varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \le x/2 \le \pi - \varepsilon$, meaning that the partial sums are uniformly bounded with respect to $x \in (0, 2\pi)$. Similarly, if $s = -1$, we would have

$$\left|\sum_{k=1}^{n}(-1)^{k-1} f(kx)\right| \le \frac{1}{\cos(x/2)} \le \frac{1}{\sin\varepsilon}$$

for $-\pi < x < \pi$, because $\cos(x/2) \ge \sin\varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-(\pi/2) + \varepsilon \le (x/2) \le (\pi/2) - \varepsilon$. Here the partial sums are uniformly bounded with respect to $x \in (-\pi, \pi)$.

We now suppose $a = 2, b = 1$. If $s = 1$, then

$$\left|\sum_{k=1}^{n} f((2k-1)x)\right| \le \frac{1}{\sin x} \le \frac{1}{\sin\varepsilon}$$

for $0 < x < \pi$, because $\sin x \ge \sin\varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \le x \le \pi - \varepsilon$. The partial sums are uniformly bounded with respect to $x \in (0, \pi)$. Finally if $s = -1$, then

$$\left|\sum_{k=1}^{n}(-1)^{k-1} f((2k-1)x)\right| \le \frac{1}{\cos x} \le \frac{1}{\sin\varepsilon}$$

for $-(\pi/2) < x < (\pi/2)$, because $\cos x \ge \sin\varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-(\pi/2) + \varepsilon \le x \le (\pi/2) - \varepsilon$, and the partial sums are uniformly bounded with respect to $x \in (-(\pi/2), (\pi/2))$. Since in each of the particular cases the sequence $1/(an-b)^\alpha$ tends to 0 for $\alpha > 0$, by virtue of Dirichlet's test all the series in question uniformly converge, whereby the proof is complete. ■

2. The general formula

THEOREM 1 *For the values of* α *and the other relevant parameters from Table* 1*, there holds*

$$\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)x)}{(an-b)^{\alpha}}=\frac{c\pi x^{\alpha-1}}{2\Gamma(\alpha)f(\pi\alpha/2)}+\sum_{k=0}^{\infty}\frac{(-1)^{k}F(\alpha-2k-\delta)}{(2k+\delta)!}x^{2k+\delta}. \tag{2}$$

Proof We make use of the *polylogarithm* $\mathrm{Li}_{\alpha}(z)$ defined by the series (see [4]), and with the following integral representation

$$\mathrm{Li}_{\alpha}(z)=\sum_{n=1}^{\infty}\frac{z^{n}}{n^{\alpha}}=\frac{1}{\Gamma(\alpha)}\int_{0}^{\infty}\frac{t^{\alpha-1}}{\mathrm{e}^{t}/z-1}\mathrm{d}t,$$

where the right-hand side integral converges for $z\in\mathbb{C}\setminus\{z\mid z\in\mathbb{R}, z\geq 1\}$, and it is referred to as *Bose's integral*. So for $\alpha>0$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sin nx}{n^{\alpha}}&=\frac{i}{2}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-inx}-\mathrm{e}^{inx}}{n^{\alpha}}=\frac{i}{2}\left(\mathrm{Li}_{\alpha}(\mathrm{e}^{-ix})-\mathrm{Li}_{\alpha}(\mathrm{e}^{ix})\right),\\ \sum_{n=1}^{\infty}\frac{\cos nx}{n^{\alpha}}&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-inx}+\mathrm{e}^{inx}}{n^{\alpha}}=\frac{1}{2}\left(\mathrm{Li}_{\alpha}(\mathrm{e}^{-ix})+\mathrm{Li}_{\alpha}(\mathrm{e}^{ix})\right)\end{aligned} \tag{3}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin nx}{n^{\alpha}}&=\frac{i}{2}\left(\mathrm{Li}_{\alpha}(-\mathrm{e}^{ix})-\mathrm{Li}_{\alpha}(-\mathrm{e}^{-ix})\right),\\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos nx}{n^{\alpha}}&=-\frac{1}{2}\left(\mathrm{Li}_{\alpha}(-\mathrm{e}^{-ix})+\mathrm{Li}_{\alpha}(-\mathrm{e}^{ix})\right).\end{aligned} \tag{4}$$

Also, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^{\alpha}}&=\frac{i}{2}\left(\left(\mathrm{Li}_{\alpha}(\mathrm{e}^{-ix})-\frac{1}{2^{\alpha}}\mathrm{Li}_{\alpha}(\mathrm{e}^{-2ix})\right)-\left(\mathrm{Li}_{\alpha}(\mathrm{e}^{ix})-\frac{1}{2^{\alpha}}\mathrm{Li}_{\alpha}(\mathrm{e}^{2ix})\right)\right),\\ \sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^{\alpha}}&=\frac{1}{2}\left(\left(\mathrm{Li}_{\alpha}(\mathrm{e}^{-ix})-\frac{1}{2^{\alpha}}\mathrm{Li}_{\alpha}(\mathrm{e}^{-2ix})\right)+\left(\mathrm{Li}_{\alpha}(\mathrm{e}^{ix})-\frac{1}{2^{\alpha}}\mathrm{Li}_{\alpha}(\mathrm{e}^{2ix})\right)\right)\end{aligned} \tag{5}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{\alpha}}&=\frac{1}{4}\left(\mathrm{Li}_{\alpha}(-i\mathrm{e}^{ix})-\mathrm{Li}_{\alpha}(-i\mathrm{e}^{-ix})-\mathrm{Li}_{\alpha}(i\mathrm{e}^{ix})+\mathrm{Li}_{\alpha}(i\mathrm{e}^{-ix})\right),\\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{\alpha}}&=\frac{i}{4}\left(\mathrm{Li}_{\alpha}(-i\mathrm{e}^{ix})+\mathrm{Li}_{\alpha}(-i\mathrm{e}^{-ix})-\mathrm{Li}_{\alpha}(i\mathrm{e}^{ix})-\mathrm{Li}_{\alpha}(i\mathrm{e}^{-ix})\right).\end{aligned} \tag{6}$$

Here, we note that in order to obtain the right-hand sides of Equation (4) and (6), we have taken advantage of the representation $(-1)^{n}=\cos n\pi$, and then, by means of elementary trigonometric identities, we split up the product into the sum of two trigonometric functions.

We shall now consider the Mellin transform of the polylogarithm in the form of Bose's integral. The Mellin transform of a function f and the inverse transform of a function φ are (see [6])

$$M(f(x)) = \int_0^\infty x^{u-1} f(x)\mathrm{d}x, \qquad M^{-1}(\varphi(u)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-u}\varphi(u)\mathrm{d}u.$$

This integral transform is closely connected to the theory of Dirichlet series and is often used in number theory and the theory of asymptotic expansions. Also, it is closely related to the Laplace and Fourier transform as well as to the theory of the gamma function and allied special functions. So we find

$$M(\mathrm{Li}_\alpha(p\,\mathrm{e}^{-x})) = \int_0^\infty x^{u-1}\mathrm{Li}_\alpha(p\,\mathrm{e}^{-x})\mathrm{d}x = \frac{1}{\Gamma(\alpha)}\int_0^\infty\int_0^\infty \frac{t^{\alpha-1}x^{u-1}}{\mathrm{e}^{t+x}/p-1}\mathrm{d}t\mathrm{d}x.$$

The change of variables $x = ab$, $t = a(1-b)$ allows the integrals to be separated

$$M(\mathrm{Li}_\alpha(p\,\mathrm{e}^{-x})) = \frac{1}{\Gamma(\alpha)}\int_0^1 b^{u-1}(1-b)^{\alpha-1}\mathrm{d}b\int_0^\infty \frac{a^{\alpha+u-1}}{\mathrm{e}^a/p-1}\mathrm{d}a = \Gamma(u)\mathrm{Li}_{\alpha+u}(p). \tag{7}$$

For $p = 1$, because $\mathrm{Li}_{\alpha+u}(1) = \zeta(\alpha+u)$, through the inverse Mellin transform, we have

$$\mathrm{Li}_\alpha(\mathrm{e}^{-x}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(u)\zeta(\alpha+u)x^{-u}\mathrm{d}u,$$

where c is a constant to the right of the poles of the integrand. The path of integration may be converted into a closed contour, and the poles of the integrand are those of $\Gamma(u)$ at $u = 0, -1, -2, \ldots$, and of $\zeta(\alpha+u)$ at $u = 1-\alpha$. Summing the residues yields a representation of the polylogarithm as a power series

$$\mathrm{Li}_\alpha(\mathrm{e}^\mu) = (-\mu)^{\alpha-1}\Gamma(1-\alpha) + \sum_{k=0}^\infty \frac{\zeta(\alpha-k)}{k!}\mu^k, \quad |\mu| < 2\pi, \quad \alpha \neq 1, 2, 3, \ldots \tag{8}$$

about $\mu = 0$. Further, following Equation (3), we have

$$\begin{aligned}\frac{i}{2}(\mathrm{Li}_\alpha(\mathrm{e}^{-\mu}) - \mathrm{Li}_\alpha(\mathrm{e}^\mu)) &= \frac{i}{2}(\mu^{\alpha-1} - (-\mu)^{\alpha-1})\Gamma(1-\alpha) - i\sum_{k=0}^\infty \frac{\zeta(\alpha-2k-1)}{(2k+1)!}\mu^{2k+1},\\ \frac{1}{2}(\mathrm{Li}_\alpha(\mathrm{e}^\mu) + \mathrm{Li}_\alpha(\mathrm{e}^{-\mu})) &= \frac{1}{2}((-\mu)^{\alpha-1} + \mu^{\alpha-1})\Gamma(1-\alpha) + \sum_{k=0}^\infty \frac{\zeta(\alpha-2k)}{(2k)!}\mu^{2k},\end{aligned} \tag{9}$$

where we replace μ with ix, $0 < x < 2\pi$. For a positive real non-integer α, the gamma function is finite, and so in view of

$$(\pm i)^{\alpha-1} = \mathrm{e}^{\pm i(\alpha-1)\pi/2} = \cos\frac{\pi}{2}(\alpha-1) \pm i\sin\frac{\pi}{2}(\alpha-1), \tag{10}$$

we calculate

$$\begin{aligned}\frac{i}{2}(\mu^{\alpha-1} - (-\mu)^{\alpha-1})\Gamma(1-\alpha) &= \frac{i\pi x^{\alpha-1}}{2\Gamma(\alpha)\sin\pi\alpha}(i^{\alpha-1} - (-i)^{\alpha-1}) = \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha)\sin(\pi/2)\alpha},\\ \frac{1}{2}((-\mu)^{\alpha-1} + \mu^{\alpha-1})\Gamma(1-\alpha) &= \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha)\sin\pi\alpha}((-i)^{\alpha-1} + i^{\alpha-1}) = \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha)\cos(\pi/2)\alpha},\end{aligned}$$

which, in conjunction with Equation (9) gives the sums of the series in Equation (3), *i.e.*

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^\alpha} = \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha)\sin(\pi/2)\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k \zeta(\alpha-2k-1)}{(2k+1)!} x^{2k+1},$$
$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha} = \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha)\cos(\pi/2)\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k \zeta(\alpha-2k)}{(2k)!} x^{2k}.$$

We now consider Equation (4). Because of Equation (7), we find $M(\mathrm{Li}_\alpha(-\mathrm{e}^{-x})) = -\Gamma(u)\eta(\alpha+u)$, since $\mathrm{Li}_{\alpha+u}(-1) = -\eta(\alpha+u)$. Through the inverse Mellin transform and conversion of path of integration into a closed contour, after summing the residues at $u = 0, -1, -2, \ldots$ and $u = 1-\alpha$ (which is 0), we have

$$\mathrm{Li}_\alpha(-\mathrm{e}^{\mu}) = -\sum_{k=0}^{\infty} \frac{\eta(\alpha-k)}{k!} \mu^k, \quad |\mu| < 2\pi, \quad \alpha \neq 1, 2, 3, \ldots,$$

whence we find

$$\begin{aligned} \frac{i}{2}(\mathrm{Li}_\alpha(-\mathrm{e}^{-\mu}) - \mathrm{Li}_\alpha(-\mathrm{e}^{\mu})) &= i\sum_{k=0}^{\infty} \frac{\eta(\alpha-2k-1)}{(2k+1)!} \mu^{2k+1}, \\ -\frac{1}{2}(\mathrm{Li}_\alpha(-\mathrm{e}^{\mu}) + \mathrm{Li}_\alpha(-\mathrm{e}^{-\mu})) &= \sum_{k=0}^{\infty} \frac{\eta(\alpha-2k)}{(2k)!} \mu^{2k}, \end{aligned} \tag{11}$$

and for $\mu = ix$ $(-\pi < x < \pi)$, we obtain the sums of the series in Equation (4), *i.e.*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nx}{n^\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k \eta(\alpha-2k-1)}{(2k+1)!} x^{2k+1},$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos nx}{n^\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k \eta(\alpha-2k)}{(2k)!} x^{2k}.$$

As regards Equation (5), judging by its structure, we have to repeat the procedure as that for obtaining Equation (8), *i.e.* once for $\mathrm{Li}_\alpha(\mathrm{e}^{\pm\mu})$ $(|\mu| < 2\pi)$, and once again for $\mathrm{Li}_\alpha(\mathrm{e}^{\pm 2\mu})$ $(|\mu| < \pi)$, so that we have

$$\begin{aligned} &\frac{i}{2}\left(\left(\mathrm{Li}_\alpha(\mathrm{e}^{-\mu}) - \frac{1}{2^\alpha}\mathrm{Li}_\alpha(\mathrm{e}^{-2\mu})\right) - \left(\mathrm{Li}_\alpha(\mathrm{e}^{\mu}) - \frac{1}{2^\alpha}\mathrm{Li}_\alpha(\mathrm{e}^{2\mu})\right)\right) \\ &\quad = \frac{i}{4}(\mu^{\alpha-1} - (-\mu)^{\alpha-1})\Gamma(1-\alpha) - i\sum_{k=0}^{\infty} \frac{\lambda(\alpha-2k-1)}{(2k+1)!} \mu^{2k+1}, \\ &\frac{1}{2}\left(\left(\mathrm{Li}_\alpha(\mathrm{e}^{-\mu}) - \frac{1}{2^\alpha}\mathrm{Li}_\alpha(\mathrm{e}^{-2\mu})\right) + \left(\mathrm{Li}_\alpha(\mathrm{e}^{\mu}) - \frac{1}{2^\alpha}\mathrm{Li}_\alpha(\mathrm{e}^{2\mu})\right)\right) \\ &\quad = \frac{1}{4}(\mu^{\alpha-1} + (-\mu)^{\alpha-1})\Gamma(1-\alpha) + \sum_{k=0}^{\infty} \frac{\lambda(\alpha-2k)}{(2k)!} \mu^{2k}, \quad |\mu| < \pi, \quad \alpha \neq 1, 2, 3, \ldots, \end{aligned} \tag{12}$$

where we have taken into account $\lambda(z) = (1 - 2^{-z})\zeta(z)$. Thus, for $\mu = ix$ $(0 < x < \pi)$, we obtain the sums of the series in Equation (5), *i.e.*

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^{\alpha}} = \frac{\pi x^{\alpha-1}}{4\Gamma(\alpha)\sin(\pi/2)\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k \lambda(\alpha-2k-1)}{(2k+1)!} x^{2k+1},$$

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{\alpha}} = \frac{\pi x^{\alpha-1}}{4\Gamma(\alpha)\cos(\pi/2)\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k \lambda(\alpha-2k)}{(2k)!} x^{2k}.$$

Finally, in the case of Equation (6), we first calculate $\mathrm{Li}_{\alpha+u}(\pm i) = 2^{-(\alpha+u)}\eta(\alpha+u) \pm i\beta(\alpha+u)$ and find $M(\mathrm{Li}_{\alpha}(\pm i\mathrm{e}^{-x})) = \Gamma(u)\left(2^{-(\alpha+u)}\eta(\alpha+u) \pm i\beta(\alpha+u)\right)$. Applying again the inverse Mellin transform and converting the path of integration into closed contour, we evaluate the residues at $u = 0, -1, -2, \ldots$ and $u = 1 - \alpha$ (which is 0), so that after their summation, a rearrangement and simplification, we obtain

$$\begin{aligned} \frac{1}{4}\left(\mathrm{Li}_{\alpha}(-i\mathrm{e}^{\mu}) - \mathrm{Li}_{\alpha}(-i\mathrm{e}^{-\mu}) - \mathrm{Li}_{\alpha}(i\mathrm{e}^{\mu}) + \mathrm{Li}_{\alpha}(i\mathrm{e}^{-\mu})\right) &= -i\sum_{k=0}^{\infty} \frac{\beta(\alpha-2k-1)}{(2k+1)!}\mu^{2k+1}, \\ \frac{i}{4}\left(\mathrm{Li}_{\alpha}(-i\mathrm{e}^{\mu}) + \mathrm{Li}_{\alpha}(-i\mathrm{e}^{-\mu}) - \mathrm{Li}_{\alpha}(i\mathrm{e}^{\mu}) - \mathrm{Li}_{\alpha}(i\mathrm{e}^{-\mu})\right) &= \sum_{k=0}^{\infty} \frac{\beta(\alpha-2k)}{(2k)!}\mu^{2k}, \end{aligned} \tag{13}$$

where $|\mu| < \pi/2, \alpha \neq 1, 2, 3, \ldots$. For $\mu = ix(-\pi/2 < x < \pi/2)$, we find the sums of the series in Equation (6), *i.e.*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(-1)^k \beta(\alpha-2k-1)}{(2k+1)!} x^{2k+1},$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(-1)^k \beta(\alpha-2k)}{(2k)!} x^{2k}.$$

Gathering all these results, we conclude that Equation (2) holds. ■

3. Closed form cases

We shall express now, for certain values of parameters, series (1) as polynomials, saying then that we have brought the infinite series in *closed form*.

THEOREM 2 *For the values of r and the other relevant parameters from Table* 1, *there holds*

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f(an-b)x)}{(an-b)^r} = \frac{(-1)^{[r/2]} c\pi x^{r-1}}{2(r-1)!} + \sum_{k=0}^{[r/2]} \frac{(-1)^k F(r-2k-\delta)}{(2k+\delta)!} x^{2k+\delta}. \tag{14}$$

Proof First of all, by setting $z = 2n + 1$ in the functional equation for the Riemann zeta function (see [1])

$$\zeta(1-z) = \frac{2\zeta(z)\Gamma(z)}{(2\pi)^z}\cos\frac{z\pi}{2},$$

we find $\zeta(-2n) = 0,\ n \in \mathbb{N}$. Taking into account the relations of the Riemann zeta functions to Dirichlet functions η and λ (see Lemma 1), we easily notice that they also vanish at even negative integers. Moreover, $\lambda(0) = 0$. It is the other way round with the Catalan function β, *i.e.* it vanishes at odd negative integers. In order to prove this, we need the Hurwitz zeta function $\zeta(z, a)$ initially defined for $\sigma > 1$ ($z = \sigma + i\tau$) by the series

$$\zeta(z, a) = \sum_{k=0}^{+\infty} \frac{1}{(k+a)^z},$$

where a is a fixed real number, $0 < a \le 1$. The function $\beta(z)$ can now be represented as follows

$$\beta(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)^z} = 4^{-z}\left(\zeta\left(z, \frac{1}{4}\right) - \zeta\left(z, \frac{3}{4}\right)\right),$$

whence we have

$$\beta(-(2n-1)) = 4^{2n-1}\left(\zeta\left(-(2n-1), \frac{1}{4}\right) - \zeta\left(-(2n-1), \frac{3}{4}\right)\right), \quad n \in \mathbb{N}.$$

Considering that for a non-negative integer n, there holds (see [1]) $\zeta(-n, a) = -B_{n+1}(a)/(n+1)$, where $B_n(x)$ are Bernoulli polynomials defined by relations

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{n=0}^{+\infty} B_n(x)\frac{t^n}{n!}, \quad B_0(x) = 1,$$

we first replace x with $1 - x$, and have

$$\sum_{n=0}^{+\infty} B_n(1-x)\frac{t^n}{n!} = \frac{t\mathrm{e}^{(1-x)t}}{\mathrm{e}^t - 1} = \frac{(-t)\mathrm{e}^{(-t)x}}{\mathrm{e}^{-t} - 1} = \sum_{n=0}^{+\infty} B_n(x)\frac{(-1)^n t^n}{n!},$$

whence we obtain $B_n(1-x) = (-1)^n B_n(x)$ implying $B_{2n}(3/4) = B_{2n}(1/4)$, so that we calculate

$$\beta(-(2n-1)) = 4^{2n-1}\left(-\frac{B_{2n}(1/4)}{2n} + \frac{B_{2n}(3/4)}{2n}\right) = 0.$$

It is known that for $r \in \mathbb{N}$, the value $\Gamma(1-r)$ of the gamma function becomes infinite, and we cannot place $\alpha = r$ immediately in Equation (9). In order to get a finite value we take, on the right-hand side sum in Equation (9), the first $m - 1$ terms if $r = 2m - 1$, and the first m terms if $r = 2m$. Then we take limits

$$\lim_{\alpha \to 2m-1}\left(\frac{i}{2}(\mu^{\alpha-1} - (-\mu)^{\alpha-1})\Gamma(1-\alpha) - i\sum_{k=0}^{m-1}\frac{\zeta(\alpha - 2k - 1)}{(2k+1)!}\mu^{2k+1}\right)$$
$$= -i\frac{\log(-\mu) - \log\mu}{2(2m-2)!}\mu^{2m-2} - i\sum_{k=0}^{m-1}\frac{\zeta(2m-2k-2)}{(2k+1)!}\mu^{2k+1},$$

$$\lim_{\alpha \to 2m}\left(\frac{1}{2}((-\mu)^{\alpha-1} + \mu^{\alpha-1})\Gamma(1-\alpha) + \sum_{k=0}^{m}\frac{\zeta(\alpha - 2k)}{(2k)!}\mu^{2k}\right)$$
$$= -\frac{\log(-\mu) - \log\mu}{2(2m-1)!}\mu^{2m-1} + \sum_{k=0}^{m}\frac{\zeta(2m-2k)}{(2k)!}\mu^{2k}.$$

By virtue of $\zeta(-2n) = 0$ $(n \in \mathbb{N})$, the remainders of the series in Equation (9) vanish, so placing $\mu = ix$, $0 < x < 2\pi$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2m-1}} &= \frac{(-1)^{m-1}\pi x^{2m-2}}{2(2m-2)!} + \sum_{k=0}^{m-1} \frac{(-1)^k \zeta(2m-2k-2)}{(2k+1)!} x^{2k+1}, \\ \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m}} &= \frac{(-1)^{m}\pi x^{2m-1}}{2(2m-1)!} + \sum_{k=0}^{m} \frac{(-1)^k \zeta(2m-2k)}{(2k)!} x^{2k} \ (m \in \mathbb{N}), \end{aligned} \tag{15}$$

which is formula (14) for $s = 1, a = 1, b = 0, c = 1, F = \zeta,\ f = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\},\ r = \left\{ \begin{smallmatrix} 2m-1 \\ 2m \end{smallmatrix} \right\}$.

When dealing with Equation (11), we do not have a problem with the singularity of the gamma function any more, so that we may replace there α with $r \in \mathbb{N}$, and considering that $\eta(-2n) = 0$ $(n \in \mathbb{N})$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nx}{n^{2m-1}} &= \sum_{k=0}^{m-1} \frac{(-1)^k \eta(2m-2k-2)}{(2k+1)!} x^{2k+1}, \\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos nx}{n^{2m}} &= \sum_{k=0}^{m} \frac{(-1)^k \eta(2m-2k)}{(2k)!} x^{2k} \ (m \in \mathbb{N}), \end{aligned} \tag{16}$$

which is formula (14) for the choice of parameters $s = -1, a = 1, b = 0, c = 0, F = \eta,\ f = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}\ r = \left\{ \begin{smallmatrix} 2m-1 \\ 2m \end{smallmatrix} \right\}$.

In the case of Equation (12), there appears again the singularity of $\Gamma(1-r)$ at $r \in \mathbb{N}$, so quite similarly as above we take limits, and after placing $\mu = ix,\ 0 < x < \pi$, because $\lambda(-2n) = 0$ $(n \in \mathbb{N})$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^{2m-1}} &= \frac{(-1)^{m-1}\pi x^{2m-2}}{4(2m-2)!} + \sum_{k=0}^{m-1} \frac{(-1)^k \lambda(2m-2k-2)}{(2k+1)!} x^{2k+1}, \\ \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2m}} &= \frac{(-1)^{m}\pi x^{2m-1}}{4(2m-1)!} + \sum_{k=0}^{m} \frac{(-1)^k \lambda(2m-2k)}{(2k)!} x^{2k} \ (m \in \mathbb{N}), \end{aligned} \tag{17}$$

which is formula (14) for the choice of $s = 1, a = 2, b = 1, c = 1/2, F = \lambda,\ f = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}\ r = \left\{ \begin{smallmatrix} 2m-1 \\ 2m \end{smallmatrix} \right\}$.

Finally, in Equation (13), similarly as for Equation (11), we do not have to deal with the singularity of the gamma function and replacing α with $r \in \mathbb{N}$, considering that $\beta(-2n+1) = 0$ $(n \in \mathbb{N})$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m}} &= \sum_{k=0}^{m} \frac{(-1)^k \beta(2m-2k-1)}{(2k+1)!} x^{2k+1}, \\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m-1}} &= \sum_{k=0}^{m-1} \frac{(-1)^k \beta(2m-2k)}{(2k)!} x^{2k} \ (m \in \mathbb{N}), \end{aligned} \tag{18}$$

which is formula (14) for the choice of parameters $s = -1, a = 2, b = 1, c = 0, F = \beta,\ f = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\} r = \left\{ \begin{smallmatrix} 2m \\ 2m-1 \end{smallmatrix} \right\}$, wherby we complete the proof. ■

We note that because of $\lambda(0) = 0$ the upper bounds in Equation (17) are actually $m-2$ and $m-1$ and $m-1$ for the first sum in Equation (18) since $\beta(-1) = 0$, but purely for the sake of fitting them into a general formula (14), we retain $m-1$ and m in Equation (17), and m for the first sum in Equation (18), without formally changing values.

3.1. *Some applications*

Now we are going to present significant and important applications of our closed form formula (14).

3.1.1. *Integral transforms*

Applying some of the integral transforms, one can obtain various series in closed form. For instance, if we take $f = \cos$, $\delta = 0$ in Equation (14), then apply the Laplace transform, we have

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} p}{(an-b)^r (p^2 + (an-b)^2)} = (-1)^{[r/2]} \frac{c\pi}{2p^r} + \sum_{i=0}^{[r/2]} \frac{(-1)^i F(r-2i)}{p^{2i+1}}.$$

Further, if we set $s = 1, a = 1, b = 0$, then $F = \zeta, c = 1$ must be taken, and we obtain

$$\sum_{n=1}^{\infty} \frac{p}{n^r (p^2 + n^2)} = (-1)^{[r/2]} \frac{\pi}{2p^r} + \sum_{i=0}^{[r/2]} \frac{(-1)^i \zeta(r-2i)}{p^{2i+1}}. \tag{19}$$

Now we apply the inverse Mellin transform to this series, knowing that (see [6, p. 166, 2.16 and p. 167, 2.25])

$$M^{-1}\left(\frac{z}{z^2+n^2}\right) = \begin{cases} \cos(n \log x) & x < 1 \\ 0 & x > 1, \end{cases} \qquad M^{-1}\left(\frac{1}{z^\nu}\right) = \begin{cases} \dfrac{(\log(1/x))^{\nu-1}}{\Gamma(\nu)} & x < 1 \\ 0 & x > 1, \end{cases}$$

coming to the sum of a trigonometric series

$$\sum_{n=1}^{\infty} \frac{\cos(n \log x)}{n^r} = \frac{(-1)^{[r/2]}\pi}{2(r-1)!} \left(\log \frac{1}{x}\right)^{r-1} + \sum_{i=0}^{[r/2]} \frac{(-1)^i \zeta(r-2i)}{(2i)!} \left(\log \frac{1}{x}\right)^{2i}, \quad x < 1.$$

However, if we want to apply the Bessel instead of Mellin transform to series (19), we first refer to (see [5, p. 36, 4.23 and p. 33, 4.6])

$$B\left(\frac{x^{\nu+1/2}}{(a^2+x^2)^\mu}\right) = \frac{a^{\nu-\mu+1}\, y^{\mu-1/2}}{2^{\mu-1}\,\Gamma(\mu)} K_{\nu-\mu+1}(ay)$$

and

$$B(x^\mu) = \frac{2^{\mu+1/2}\,\Gamma((\nu/2)+(\mu/2)+(3/4))}{y^{\mu+1}\,\Gamma((\nu/2)-(\mu/2)+(1/4))},$$

where K_ν is Hankel's function. Thus we obtain the sum of one of the Schlömilch series

$$\sum_{n=1}^{\infty} \frac{K_{1/2}(ny)}{n^{r-(1/2)}} = \frac{(-1)^{[r/2]}\pi y^{r-(3/2)}\Gamma(1-(r/2))}{2^{r+1/2}\Gamma((r+1)/2)} + \sum_{i=0}^{[r/2]} \frac{(-1)^i \zeta(r-2i)\Gamma((1/2)-i)}{i!\, 2^{2i+1/2}}\, y^{2i-1/2}.$$

3.1.2. *Convergence acceleration*

On the basis of Equation (14), the convergence acceleration of trigonometric series is obtained. As an example, we consider the series

$$T = \sum_{n=1}^{\infty} \frac{n}{n^2+1} \sin nx. \tag{20}$$

For each natural number M we prove, by the method of mathematical induction, that there holds

$$\begin{aligned} \frac{n}{n^2+1} &= \frac{n}{n^2} \cdot \frac{1}{1+(1/n^2)} = \frac{1}{n}\left(1 - \frac{1}{n^2} + \frac{1}{n^4} - \cdots\right) \\ &= \sum_{m=1}^{M} \frac{(-1)^{m-1}}{n^{2m-1}} + \frac{(-1)^M}{n^{2M-1}(n^2+1)}. \end{aligned} \tag{21}$$

Replacing Equation (21) in Equation (20), we have

$$T = \sum_{m=1}^{M} (-1)^{m-1} T_{2m-1}^{\sin} + \sum_{n=1}^{\infty} \frac{(-1)^M \sin nx}{(n^2+1)n^{2M-1}},$$

where $T_{2m-1}^{\sin}$ denotes first of the closed form formulas (15), *i.e.*

$$T_{2m-1}^{\sin} = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2m-1}} = \frac{(-1)^{m-1}\pi x^{2m-2}}{2(2m-2)!} + \sum_{k=0}^{m-1} \frac{(-1)^k \zeta(2m-2k-2)}{(2k+1)!} x^{2k+1}.$$

So, the greater M, the faster convergence of the remaining series, and accordingly we have the faster convergence of the series T. Here is shown how many terms of the remaining series ought to be taken for the given M and accuracy ε

ε	10^{-1}	10^{-2}	10^{-5}	10^{-8}
$M = 1$	3	8	224	7072
$M = 3$	2	2	6	16
$M = 10$	1	2	2	3

4. Limiting values

THEOREM 3 *For the values of* α *complementary to those in Table* 1, *there holds*

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^{\alpha}} = \Phi_{\alpha}(x) + \sum_{k=[(\alpha-1)/2]+1}^{\infty} \frac{(-1)^k F(\alpha-2k-\delta)}{(2k+\delta)!} x^{2k+\delta}, \tag{22}$$

where

$$\Phi_{\alpha}(x) = \frac{(-1)^{[(\alpha-1)/2]} c}{(\alpha-1)!}(\psi(\alpha)+\gamma-\log x)\, x^{\alpha-1} + \sum_{k=1}^{[(\alpha-1)/2]} \frac{(-1)^{[(\alpha-1)/2]-k} F(2k+1)}{(\alpha-2k-1)!} x^{\alpha-2k-1}$$

and if $c = 1$, *then* $F = \zeta$ *or if* $c = 1/2$, *then* $F = \lambda$.

Proof We have seen that if we let α tend to $2m-1$ and to $2m$, respectively, in Equation (9), the right-hand side sums truncate because the zeta function vanish at even negative integers, and as limiting values, we obtain polynomials (see Equation (15)). However, if we let α tend to $2m$ and to $2m-1$, respectively in Equation (9), we find

$$\lim_{\alpha\to 2m}\left(\frac{i}{2}(\mu^{\alpha-1}-(-\mu)^{\alpha-1})\Gamma(1-\alpha)-i\sum_{k=0}^{m-1}\frac{\zeta(\alpha-2k-1)}{(2k+1)!}\mu^{2k+1}\right)$$
$$=\frac{i}{(2m-1)!}\left(\psi(2m)+\gamma-\frac{\log(-\mu)+\log\mu}{2}\right)\mu^{2m-1}-i\sum_{k=1}^{m-1}\frac{\zeta(2k+1)}{(2m-2k-1)!}\mu^{2m-2k-1}$$
$$\lim_{\alpha\to 2m-1}\left(\frac{1}{2}((-\mu)^{\alpha-1}+\mu^{\alpha-1})\Gamma(1-\alpha)+\sum_{k=0}^{m-1}\frac{\zeta(\alpha-2k)}{(2k)!}\mu^{2k}\right)$$
$$=\frac{1}{(2m-2)!}\left(\psi(2m-1)+\gamma-\frac{\log(-\mu)+\log\mu}{2}\right)\mu^{2m-2}+\sum_{k=1}^{m-1}\frac{\zeta(2k+1)}{(2m-2k-2)!}\mu^{2m-2k-2},$$

where ψ is the digamma function. Setting $\mu=ix$, $0<x<2\pi$, we obtain

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2m}}=\frac{(-1)^{m-1}}{(2m-1)!}(\psi(2m)+\gamma-\log x)\,x^{2m-1}+\sum_{k=1}^{m-1}\frac{(-1)^{m-1-k}\zeta(2k+1)}{(2m-2k-1)!}x^{2m-2k-1}+\sum_{k=m}^{\infty}\frac{(-1)^k\zeta(2m-2k-1)}{(2k+1)!}x^{2k+1},$$
$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2m-1}}=\frac{(-1)^{m-1}}{(2m-2)!}(\psi(2m-1)+\gamma-\log x)\,x^{2m-2}+\sum_{k=1}^{m-1}\frac{(-1)^{m-1-k}\zeta(2k+1)}{(2m-2k-2)!}x^{2m-2k-2}+\sum_{k=m}^{\infty}\frac{(-1)^k\zeta(2m-2k-1)}{(2k)!}x^{2k},\tag{23}$$

which is Equation (22) for $s=1, a=1, b=0, c=1, F=\zeta, f=\left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}\ \alpha=\left\{\begin{smallmatrix}2m\\ 2m-1\end{smallmatrix}\right\}\ \delta=\left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$.

In the case of Equation (12), bearing in mind $\lambda(z)=(1-2^{-z})\zeta(z)$, and applying the same procedure as above yields

$$\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^{2m}}=\frac{(-1)^{m-1}}{2(2m-1)!}(\psi(2m)+\gamma-\log x)\,x^{2m-1}+\sum_{k=1}^{m-1}\frac{(-1)^{m-1-k}\lambda(2k+1)}{(2m-2k-1)!}x^{2m-2k-1}+\sum_{k=m}^{\infty}\frac{(-1)^k\lambda(2m-2k-1)}{(2k+1)!}x^{2k+1},$$
$$\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^{2m-1}}=\frac{(-1)^{m-1}}{2(2m-2)!}(\psi(2m-1)+\gamma-\log x)\,x^{2m-2}+\sum_{k=1}^{m-1}\frac{(-1)^{m-1-k}\lambda(2k+1)}{(2m-2k-2)!}x^{2m-2k-2}+\sum_{k=m}^{\infty}\frac{(-1)^k\lambda(2m-2k-1)}{(2k)!}x^{2k},$$

which is Equation (22) for $s=1, a=2, b=1, c=1/2, F=\lambda, f=\left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}\ \alpha=\left\{\begin{smallmatrix}2m\\ 2m-1\end{smallmatrix}\right\}\ \delta=\left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$. ■

Remark 2 We note that formulas (23) can be obtained by letting in Equation (2) α tend to $2m$ if $f = \sin$ or to $2m - 1$ if $f = \cos$ ($m \in \mathbb{N}$), since we encounter division by zero. Apart from this, the left-hand side series are known as *Clausen functions* defined by (see [4])

$$\mathrm{Cl}_{2\nu}(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2\nu}}, \quad \mathrm{Cl}_{2\nu-1}(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2\nu-1}}, \quad \nu \in \mathbb{N},$$

and, as a by-product, we have managed to express each of them as the sum of a polynomial and a power series in terms of Riemann's ζ function, that is

$$\mathrm{Cl}_{2\nu}(x) = \frac{(-1)^{\nu-1}}{(2\nu-1)!}\,(\psi(2\nu) + \gamma - \log x)\, x^{2\nu-1} + \sum_{k=1}^{\nu-1} \frac{(-1)^{\nu-1-k}\zeta(2k+1)}{(2\nu-2k-1)!}\, x^{2\nu-2k-1}$$
$$+ \sum_{k=\nu}^{\infty} \frac{(-1)^k \zeta(2\nu-2k-1)}{(2k+1)!}\, x^{2k+1}, \quad \nu \in \mathbb{N},$$

$$\mathrm{Cl}_{2\nu-1}(x) = \frac{(-1)^{\nu-1}}{(2\nu-2)!}\,(\psi(2\nu-1) + \gamma - \log x)\, x^{2\nu-2} + \sum_{k=1}^{\nu-1} \frac{(-1)^{\nu-1-k}\zeta(2k+1)}{(2\nu-2k-2)!}\, x^{2\nu-2k-2}$$
$$+ \sum_{k=\nu}^{\infty} \frac{(-1)^k \zeta(2\nu-2k-1)}{(2k)!}\, x^{2k}, \quad \nu \in \mathbb{N}.$$

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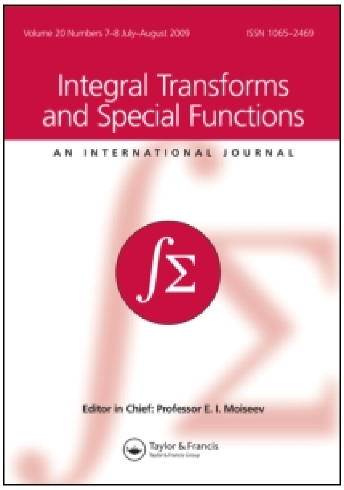

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On trigonometric series over integrals involving Bessel or Struve functions

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On trigonometric series over integrals involving Bessel or Struve functions

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To find formulas for the summation of trigonometric series over integrals involving Bessel or Struve functions, we rely on trigonometric series involving Bessel or Struve functions, which are in turn obtained by using summation formulas for series over the product of two trigonometric functions. All these sums are expressed either as power series in terms of Riemann's ζ or Catalan's β function or Dirichlet functions η and λ, or, in certain cases, they are brought in so called closed form, which means that the infinite series are represented by finite sums. Important limiting values cases are considered too.

Keywords: Riemann's; Catalan's function; Bessel; Struve and Dirichlet functions

MSC (2000): Primary: 33C10; Secondary: 11M06; 65B10

1. Introduction

In this article, we deal with finding the sum of the series

$$I_{\alpha}^{D,f} = \sum_{n=1}^{\infty} \frac{(s)^{n-1} D_{\nu}((an-b)x)}{(an-b)^{\alpha}} f((an-b)z), \qquad (1)$$

where $a = \left\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}$ $b = \left\{ \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right\}$, $s = 1$ or -1, $\alpha \in \mathbb{R}^{+}$, $f = \sin$ or $f = \cos$, and, $D_{\nu}(x)$ denotes an integral $B_{\nu,\phi}(x)$ or $S_{\nu,\phi}$, defined by

$$B_{\nu,\phi}(x) = \int_0^1 J_{\nu}(xy)\phi(y)\mathrm{d}y, \quad S_{\nu,\phi}(x) = \int_0^1 \mathbf{H}_{\nu}(xy)\phi(y)\mathrm{d}y, \qquad (2)$$

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Table 1. Parameters and convergence regions.

a	b	s	c	F	for
1	0	1	1	ζ	$0<x<2\pi$
		-1	0	η	$-\pi<x<\pi$
2	1	1	$\frac{1}{2}$	λ	$0<x<\pi$
		-1	0	β	$-\frac{\pi}{2}<x<\frac{\pi}{2}$

where J_ν and $\mathbf{H}_\nu$ are Bessel or Struve functions of the first kind and order $\nu \in R$. To obtain the sum of the series (1) *we do not have to previously calculate integrals* $D_\nu((an-b)x)$, which are not necessarily found elementarily. However, if we are able to do so, the series (1) will take a different form, leading possibly to a new class of summation formulas. At first, we assume that ϕ is integrable. Yet, in order to extend the class of summable series, we admit that ϕ is differentiable on $(0,1)$, but not bounded in the neighbourhood of 0 or 1, or even not integrable on $(0,1)$, however, such that there exists at least one of the integrals (2). We further require that the functions $y^k\phi(y)$ $(k\in\mathbb{N})$ are integrable on $(0,1)$ as well.

Obtaining the sum of the series (1) relies on the summation of some trigonometric series (see [6]) in terms of Riemann's ζ or Catalan's β function or Dirichlet functions η and λ, in the form of a single formula, i.e.

$$\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)x)}{(an-b)^\alpha}=\frac{c\pi}{2\Gamma(\alpha)f(\pi\alpha/2)}x^{\alpha-1}+\sum_{i=0}^{\infty}\frac{(-1)^iF(\alpha-2i-\delta)}{(2i+\delta)!}x^{2i+\delta}, \tag{3}$$

where $\alpha>0$, $a=\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}$ $b=\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}$, $s=1$ or -1, and $f=\left\{\begin{smallmatrix}\sin\\\cos\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\}$. The values for F and c are in the Table 1, where ζ is Riemann's zeta function $\zeta(z)=\sum_{k=1}^{\infty}k^{-z}$, η and λ are Dirichlet functions $\eta(z)=\sum_{k=1}^{\infty}(-1)^{k-1}k^{-z}=(1-2^{1-z})\zeta(z)$, $\lambda(z)=\sum_{k=0}^{\infty}(2k+1)^{-z}=(1-2^{-z})\zeta(z)$ and β is Catalan's function $\beta(z)=\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-z}$.

We note that the functions ζ,η,λ are analytic in the whole complex plane except for $z=1$, where they have a pole. The integral representation $\beta(z)=1/\Gamma(z)\int_0^\infty(x^{z-1}\mathrm{e}^x)/(\mathrm{e}^{2x}+1)\,\mathrm{d}x$ of Catalan's function defines an analytical function for $\Re z\geq 1$; but, also it satisfies the functional equation $\beta(z)=(\pi/2)^{z-1}\Gamma(1-z)\cos(\pi z/2)\ \beta(1-z)$ extending beta to the left side of the complex plane $\Re z<1$.

2. Preliminaries

We place the integral $D(x)$ in Equation (1), and check whether the interchange of summation and integration

$$\begin{aligned} I_\alpha^{D,f}&=\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)z)}{(an-b)^\alpha}\int_0^1\varphi_\nu((an-b)xy)\,\phi(y)\mathrm{d}y\\ &=\int_0^1\left(\sum_{n=1}^{\infty}\frac{(s)^{n-1}\varphi_\nu((an-b)xy)}{(an-b)^\alpha}f((an-b)z)\right)\phi(y)\mathrm{d}y\end{aligned} \tag{4}$$

may take place. Here is $\alpha > 0$, $a = \left\{ {1 \atop 2} \right\}$ $b = \left\{ {0 \atop 1} \right\}$, $s = 1$ or -1, $f =$ sin or cos, and φ_ν is a Bessel J_ν or Struve function $\mathbf{H}_\nu$. In order to justify Equation (4), we prove uniform convergence of the series

$$S_\alpha^{\varphi, f} = \sum_{n=1}^{\infty} \frac{(s)^{n-1}\varphi_\nu((an-b)xy) f((an-b)z)}{(an-b)^\alpha} \tag{5}$$

with respect to $y \in (0, 1)$. For this purpose, we use an integral representation of the Bessel or Struve function [1]

$$\varphi_\nu(t) = \frac{2(t/2)^\nu}{\Gamma(1/2)\Gamma(\nu + 1/2)} \int_0^{\pi/2} \sin^{2\nu}\, \theta g(t\cos\theta)\, \mathrm{d}\theta, \tag{6}$$

where $\nu > -1/2$, $\varphi_\nu = \left\{ {J_\nu \atop \mathbf{H}_\nu} \right\}$ $g = \left\{ {\cos \atop \sin} \right\}$. After replacing φ_ν in Equation (5) with the right-hand side integral of Equation (6), we shall first prove that we may interchange integration and summation, i.e.

$$\begin{aligned} S_\alpha^{\varphi, f} &= \frac{2(xy/2)^\nu}{\sqrt{\pi}\Gamma(\nu + (1/2))} \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z)}{(an-b)^{\alpha-\nu}} \int_0^{\pi/2} \sin^{2\nu}\, \theta\ g((an-b)xy\cos\theta)\, \mathrm{d}\theta \\ &= \frac{2(xy/2)^\nu}{\sqrt{\pi}\Gamma(\nu + (1/2))} \int_0^{\pi/2} \sin^{2\nu}\, \theta \left(\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z)}{(an-b)^{\alpha-\nu}} g((an-b)xy\cos\theta) \right) \mathrm{d}\theta, \end{aligned} \tag{7}$$

by showing uniform convergence of the right-hand series with respect to $(y, \theta) \in (0, 1) \times [0, \pi/2]$ on the basis of Dirichlet's test, which says that (see [3]) the series $\sum_{n=0}^{\infty} a_n(y) b_n(y)$ is uniformly convergent in D, if the partial sums of $\sum_{n=0}^{\infty} a_n(y)$ are uniformly bounded in D and the sequence $b_n(y)$, being monotonic for every fixed y, uniformly converges to 0.

LEMMA 1 *The series* (5) *converges uniformly with respect to y on* $(0, 1)$.

Proof In order to prove this, we treat the right-hand series of Equation (7) as a function of $y \in (0, 1)$, regarding x, z and θ as variable parameters. By making use of an elementary trigonometric identity, the product of f and g is represented as a sum of two trigonometric functions sin or cos. Consequently, the series in question is split up into two series of the type

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1}\tau((an-b)(z \pm xy\cos\theta))}{(an-b)^{\alpha-\nu}}, \tag{8}$$

where $\tau =$ sin or $\tau =$ cos. Let us suppose first that $a = 1, b = 0$. If $s = 1$, then there holds

$$\left| \sum_{k=1}^{n} \tau(k(z \pm xy\cos\theta)) \right| \le \frac{1}{\sin(z \pm xy\cos\theta)/2} \le \frac{1}{\sin\varepsilon}$$

for $0 < z \pm xy\ \cos\ \theta < 2\pi$, because $\sin\,(z \pm xy\ \cos\ \theta)/2 \ge \sin\ \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \le (z \pm xy\ \cos\ \theta)/2 \le \pi - \varepsilon$. Similarly, if $s = -1$, we would have

$$\left| \sum_{k=1}^{n} (-1)^{k-1}\tau(k(z \pm xy\cos\theta)) \right| \le \frac{1}{\cos(z \pm xy\cos\theta)/2} \le \frac{1}{\sin\varepsilon}$$

for $-\pi < z \pm xy\ \cos\ \theta < \pi$, because $\cos\,(z \pm xy\ \cos\ \theta)/2 \ge \sin\ \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-\pi/2 + \varepsilon \le (z \pm xy\ \cos\ \theta)/2 \le \pi/2 - \varepsilon$.

We now suppose $a=2, b=1$. If $s=1$, then

$$\left|\sum_{k=1}^{n}\tau((2k-1)(z\pm xy\cos\theta))\right|\le\frac{1}{\sin(z\pm xy\cos\theta)}\le\frac{1}{\sin\varepsilon}$$

for $0<z\pm xy\ \cos\ \theta<\pi$, because $\sin(z\pm xy\ \cos\ \theta)\ge\sin\ \varepsilon>0$ for each $\varepsilon>0$ satisfying $\varepsilon\le z\pm xy\ \cos\ \theta\le\pi-\varepsilon$. Finally if $s=-1$, then

$$\left|\sum_{k=1}^{n}(-1)^{k-1}\tau((2k-1)(z\pm xy\cos\theta))\right|\le\frac{1}{\cos(z\pm xy\cos\theta)}\le\frac{1}{\sin\varepsilon}$$

for $-\pi/2<z\pm xy\ \cos\ \theta<\pi/2$, because $\cos(z\pm xy\ \cos\ \theta)\ge\sin\ \varepsilon>0$ for each $\varepsilon>0$ satisfying $-\pi/2+\varepsilon\le z\pm xy\ \cos\ \theta\le\pi/2-\varepsilon$.

On the one hand, all the partial sums are uniformly bounded with respect to $(y,\theta)\in[0,1]\times[0,\pi/2]$, and considering that $0\le y\ \cos\ \theta\le 1$, we are able to determine boundaries for x and z, i.e. the convergence regions for the series (1). They are in Table 2. For instance, in the first case, from $0\le y\ \cos\ \theta\le 1$, we have immediately $-|x|\le\pm xy\ \cos\ \theta\le|x|$, and to come to the condition $0<z\pm xy\ \cos\ \theta<2\pi$, it is necessary to take $|x|<z<2\pi-|x|$; hence, there follows $|x|<\pi$. In a similar way the rest of the convergence regions are determined. Note that the boundaries for x are the same as for $xy\ \cos\ \theta$, because $0\le y\ \cos\ \theta\le 1$.

On the other hand, it is obvious that in each case the sequence $1/(an-b)^{\alpha-\nu}$ monotonically tends to 0 for $\alpha>\nu$, which proves uniform convergence of the right-hand series in Equation (7) with respect to $(y,\theta)\in(0,1)\times[0,\pi/2]$. So the interchange of integration and summation in Equation (7) is permitted. There still remains to prove that the left-hand series in Equation (7) uniformly converges with respect to $y\in(0,1)$. Namely, relying on Equation (7) and uniform convergence of the right-hand series in Equation (7) with respect to $(y,\theta)\in(0,1)\times[0,\pi/2]$, we state that for an arbitrary $\varepsilon>0$, there exists k_0, so that $k\ge k_0$ implies

$$\begin{aligned}&\left|\frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+(1/2))}\sum_{n=k+1}^{\infty}\frac{(s)^{n-1}f((an-b)z)}{(an-b)^{\alpha-\nu}}\int_0^{\pi/2}\sin^{2\nu}\theta\ g((an-b)xy\cos\theta)\,\mathrm{d}\theta\right|\\&=\left|\int_0^{\pi/2}\sin^{2\nu}\theta\left(\frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+(1/2))}\sum_{n=k+1}^{\infty}\frac{(s)^{n-1}f((an-b)z)}{(an-b)^{\alpha-\nu}}g((an-b)\,xy\cos\theta)\right)\mathrm{d}\theta\right|\\&\le\int_0^{\pi/2}\sin^{2\nu}\theta\left|\frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+(1/2))}\sum_{n=k+1}^{\infty}\frac{(s)^{n-1}f((an-b)z)}{(an-b)^{\alpha-\nu}}g((an-b)xy\cos\theta)\right|\mathrm{d}\theta<\varepsilon.\end{aligned}\tag{9}$$

Table 2. Parameters and convergence regions.

a	b	s	c	F	Convergence region
1	0	1	1	ζ	$K_1=\{(x,z)\mid-\pi<x<\pi, \lvert x\rvert<z<2\pi-\lvert x\rvert\}$
1	0	-1	0	η	$K_2=\{(x,z)\mid-\pi<x<\pi,\ \lvert x\rvert-\pi<z<\pi-\lvert x\rvert\}$
2	1	1	$\frac{1}{2}$	λ	$K_3=\{(x,z)\mid-\frac{\pi}{2}<x<\frac{\pi}{2}, \lvert x\rvert<z<\pi-\lvert x\rvert\}$
2	1	-1	0	β	$K_4=\{(x,z)\mid-\frac{\pi}{2}<x<\frac{\pi}{2}, \lvert x\rvert-\frac{\pi}{2}<z<\frac{\pi}{2}-\lvert x\rvert\}$

Hence, we conclude that the convergence speed of the first series in Equation (9) does not depend on y, meaning that the left-hand side series in Equation (7) uniformly converges with respect to $y \in (0,1)$, and so does the series (5). ■

LEMMA 2 *Let* $\phi(y)$ *be integrable. Then there holds Equation* (4).

Proof Let us denote by

$$S_{\alpha,k}^{\varphi,f} = \sum_{n=1}^{k} \frac{(s)^{k-1}\varphi_\nu((an-b)xy)f((an-b)z)}{(an-b)^\alpha} \tag{10}$$

the kth partial sum of the series (5). We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(s)^{n-1}f((an-b)z)}{(an-b)^\alpha} \int_0^1 \varphi_\nu((an-b)xy)\phi(y)\,\mathrm{d}y \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} \frac{(s)^{n-1}f((an-b)z)}{(an-b)^\alpha} \int_0^1 \varphi_\nu((an-b)xy)\phi(y)\,\mathrm{d}y \\
&= \lim_{k\to\infty} \int_0^1 \left(\sum_{n=1}^{k} \frac{(s)^{n-1}\varphi_\nu((an-b)xy)f((an-b)z)}{(an-b)^\alpha}\right)\phi(y)\,\mathrm{d}y \\
&= \int_0^1 \left(\sum_{n=1}^{\infty} \frac{(s)^{n-1}\varphi_\nu((an-b)xy)f((an-b)z)}{(an-b)^\alpha}\right)\phi(y)\,\mathrm{d}y.
\end{aligned}$$

The last passage is permitted because both the sequence (10) uniformly converges with respect to y on $(0,1)$ (Lemma 1) and $\int_0^1 \phi(y)\mathrm{d}y$ exists. ■

LEMMA 3 *Suppose that, for a differentiable on* $(0,1)$ *and unbounded in the neighbourhood of 0 or 1 function* ϕ*, the integral* $\int_0^1 \phi(y)\mathrm{d}y$ *does not converge. Let there exist at least one of the integrals* $D((an-b)x)$ *(*D *is* B_ϕ *or* S_ϕ *defined by Equation* (2)*), so that* $|D((an-b)x)| \le M_n(x)$*, and for each corresponding* x *from Table* 2*, the sequence* $M_n(x)/(an-b)^\alpha$*,* $\alpha > 0$*, monotonically tends to 0. Then there holds Equation* (4).

Proof Because of the assumption that ϕ is a differentiable function, we know that it is continuous on each closed interval within $(0,1)$, and, as it is not bounded in the neighbourhood of 0 or 1, without loss of generality, we can consider $[\delta, 1-\delta]$, $0 < \delta < 1$. Continuous function on a closed interval is bounded, so referring again to Dirichlet's test, we prove uniform convergence of the series (5) with respect to y, but this time on $[\delta, 1-\delta]$, so that there holds

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(s)^{n-1}f((an-b)z)}{(an-b)^\alpha} \int_\delta^{1-\delta} \varphi_\nu((an-b)xy)\,\phi(y)\,\mathrm{d}y \\
&= \int_\delta^{1-\delta} \left(\sum_{n=1}^{\infty} \frac{(s)^{n-1}f((an-b)z)\varphi_\nu((an-b)xy))}{(an-b)^\alpha}\right)\phi(y)\,\mathrm{d}y \quad (\alpha > 0).
\end{aligned} \tag{11}$$

We regard the left-hand side series as a function of δ with variable parameters z and x. In view of the conditions, by virtue of Dirichlet's test, the left-hand side series in Equation (11) converges uniformly with respect to δ on $(0, 1)$. Hence, we have

$$\lim_{\delta\to 0+}\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)z)}{(an-b)^{\alpha}}\int_{\delta}^{1-\delta}\varphi_{\nu}((an-b)xy)\,\phi(y))\,\mathrm{d}y$$
$$=\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)z)}{(an-b)^{\alpha}}\int_{0}^{1}\varphi_{\nu}((an-b)xy)\,\phi(y))\,\mathrm{d}y,$$

meaning that the right-hand side integral in Equation (11) converges, so there holds Equation (4). ■

3. Sum of the series over the product of a Bessel and a trigonometric function

The series involving the product of sine or cosine play a key role in finding the summation formula for the series (1). That is why we investigated them thoroughly in [5]. After representing the product of g and f as the sum of two trigonometric functions sin or cos, and applying Equation (3) to both of series in each of the particular cases, in [5] we obtained a general formula

$$\begin{aligned} T_{\alpha}^{f,g} &= \sum_{n=1}^{\infty}\frac{(s)^{n-1}g((an-b)xy\cos\theta)\,f((an-b)z)}{(an-b)^{\alpha-\nu}} \\ &= \frac{c\pi(-1)^{\delta(\delta-d)}}{4\Gamma(\alpha-\nu)h(\pi(\alpha-\nu)/2)}((z+xy\cos\theta)^{\alpha-\nu-1}+(-1)^{\delta}(z-xy\cos\theta)^{\alpha-\nu-1}) \\ &\quad+\sum_{i=0}^{\infty}\frac{(-1)^{\delta(\delta-d)+i}F(\alpha-\nu-2i-d)}{(2i+d)!}\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}(xy\cos\theta)^{2j+\delta}, \end{aligned} \tag{12}$$

where $g=\left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$ and $d=\left\{\begin{smallmatrix}0 & f=g\\ 1 & f\neq g\end{smallmatrix}\right.$, $h=\left\{\begin{smallmatrix}\cos & f=g\\ \sin & f\neq g\end{smallmatrix}\right.$. All the other relevant parameters are in Table 2.

When on the right-hand side of Equation (12) appears $h=\sin$ and $\alpha-\nu=2m$ or $h=\cos$ and $\alpha-\nu=2m-1$, where $m\in\mathbb{N}$, one should take limit. However, if $\alpha-\nu-d=2m$ and $F=\zeta,\eta,\lambda$ or $\alpha-\nu-d=2m-1$ and $F=\beta$ $(m\in\mathbb{N})$, the sum of the series on the right-hand side of Equation (12) consists of a finite number of terms because of the vanishing functions ζ,η,λ at even negative integers, and the function β at odd negative integers. So, for this choice of parameters, the formula (12) is brought into so called closed form (see [5])

$$\begin{aligned} T_{2m+d+\varepsilon}^{f,g} &= \sum_{n=1}^{\infty}\frac{(s)^{n-1}g((an-b)xy\cos\theta)\,f((an-b)z)}{(an-b)^{2m+d+\varepsilon}} \\ &= \frac{c\pi}{2}\sum_{j=0}^{m}\binom{2m+d+\varepsilon-1}{2j+\delta}\frac{(-1)^{\delta(\delta-d)}z^{2m-2j+d-\delta+\varepsilon-1}(xy\cos\theta)^{2j+\delta}}{(2m+d+\varepsilon-1)!\,h(m\pi+(d+\varepsilon)\pi/2)} \\ &\quad+\sum_{i=0}^{m}\frac{(-1)^{\delta(\delta-d)+i}F(2m-2i+\varepsilon)}{(2i+d)!}\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}(xy\cos\theta)^{2j+\delta}, \end{aligned} \tag{13}$$

where $g=\left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$, and independently of that $d=\left\{\begin{smallmatrix}0 & f=g\\ 1 & f\neq g\end{smallmatrix}\right.$, $h=\left\{\begin{smallmatrix}\cos & f=g\\ \sin & f\neq g\end{smallmatrix}\right.$. For $F=\zeta,\eta,\lambda$, there holds $\varepsilon=0$, but for $F=\beta$ it is $\varepsilon=1$. The other relevant parameters are in Table 2. The formula (13) comprises some particular results from [4], but it is more suitable for immediate obtaining sums of some infinite series as well.

Now, we find the sum (5) by virtue of the summation formula (12), replacing it in Equation (7) and developing the binomials $(z \pm xy\cos\theta)^{\alpha-\nu-1}$ into binomial series. After a rearrangement we obtain

$$S_\alpha^{\varphi,f} = \frac{(xy/2)^\nu}{G_\nu}\left(\frac{c\pi(-1)^{\delta(\delta-d)}}{2\Gamma(\alpha-\nu)h(\pi(\alpha-\nu)/2)}\sum_{j=0}^{\infty}\binom{\alpha-\nu-1}{2j+\delta}z^{\alpha-\nu-1-2j-\delta}(xy)^{2j+\delta}I_{2\nu,2j+\delta}\right.$$
$$\left.+\sum_{i=0}^{\infty}\frac{(-1)^{\delta(\delta-d)+i}F(\alpha-\nu-2i-d)}{(2i+d)!}\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}(xy)^{2j+\delta}I_{2\nu,2j+\delta}\right), \tag{14}$$

where, for the sake of simplicity, $G_\nu = \sqrt{\pi}\,\Gamma(\nu+1/2)$ and $I_{2\nu,2j+\delta} = \int_0^{\pi/2}\sin^{2\nu}\theta\,\cos^{2j+\delta}\theta\,\mathrm{d}\theta$. Introducing $\sin\theta = t$ in the last integral, we have

$$I_{2\nu,2j+\delta} = \frac{1}{2}\int_0^1 (t^2)^{(2\nu+1)/2-1}(1-t^2)^{(2j+\delta+1)/2-1}\,\mathrm{d}(t^2) = \frac{1}{2}\mathrm{B}\left(\nu+\frac{1}{2}, j+\frac{\delta+1}{2}\right), \tag{15}$$

and come to the required summation formula for the series (5)

$$S_\alpha^{\varphi,f} = \frac{(-1)^{\delta(\delta-d)}c\sqrt{\pi}(xy/2)^\nu}{2\Gamma(\alpha-\nu)h(\pi(\alpha-\nu)/2)}\sum_{j=0}^{\infty}\binom{\alpha-\nu-1}{2j+\delta}z^{\alpha-\nu-2j-1-\delta}(xy)^{2j+\delta}G_j$$
$$+\frac{(xy/2)^\nu}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{(-1)^{\delta(\delta-d)+i}F(\alpha-\nu-2i-d)}{(2i+d)!}\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}(xy)^{2j+\delta}G_j, \tag{16}$$

where $\alpha > \nu > -1/2$, $\varphi_\nu = \left\{\begin{smallmatrix} J_\nu \\ \mathbf{H}_\nu \end{smallmatrix}\right\}$ $g = \left\{\begin{smallmatrix} \cos \\ \sin \end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right\}$; independently of that $d = \left\{\begin{smallmatrix} 0 & f=g \\ 1 & f\neq g \end{smallmatrix}\right.$ and $h = \left\{\begin{smallmatrix} \cos & f=g \\ \sin & f\neq g \end{smallmatrix}\right.$. The other relevant parameters are given in Table 2, and for the sake of brevity, we have introduced $G_j = \Gamma(j+(\delta+1)/2)/\Gamma(j+\nu+1+\delta/2)$.

3.1. *Limiting values*

We shall now consider some important particular cases of the formula (16). If $h = \sin$ and $\alpha-\nu = 2m$ or $h = \cos$ and $\alpha-\nu = 2m-1$, $m \in \mathbb{N}$ division by zero is not defined, and we have to take limit. After choosing $a=1, b=0, s=1$, there must be $c=1$, $F=\zeta$. Also if we set $\varphi_\nu = J_\nu$, $g = \cos$, $\delta = 0$, $f = \cos$, then $d = 0$, $h = \cos$, and we have

$$S_\alpha^{J,\cos} = \frac{\sqrt{\pi}(xy/2)^\nu z^{\alpha-\nu-1}}{2\Gamma(\alpha-\nu)\cos(\pi(\alpha-\nu)/2)}\sum_{j=0}^{\infty}\binom{\alpha-\nu-1}{2j}\frac{(xy/z)^{2j}\Gamma(j+1/2)}{\Gamma(j+\nu+1)}$$
$$+\frac{(xy/2)^\nu}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{(-z^2)^i\zeta(\alpha-\nu-2i)}{(2i)!}\sum_{j=0}^{i}\frac{(xy/z)^{2j}\binom{2i}{2j}\Gamma(j+1/2)}{\Gamma(j+\nu+1)}.$$

In order to take limit, we first denote $\sigma=\alpha-\nu$ and replace α with $\sigma+\nu$, then we find

$$\Phi_{2m-1,\nu}(x,y,z)=\lim_{\sigma\to 2m-1}\left[\frac{\sqrt{\pi}(xy/2)^{\nu}z^{\sigma-1}}{2\Gamma(\sigma)\cos(\pi\sigma/2)}\sum_{j=0}^{m-1}\binom{\sigma-1}{2j}\frac{(xy/z)^{2j}\Gamma(j+1/2)}{\Gamma(j+\nu+1)}\right.$$
$$\left.+\frac{(xy/2)^{\nu}}{\sqrt{\pi}}\sum_{i=0}^{m-1}\frac{(-z^2)^i\zeta(\sigma-2i)}{(2i)!}\sum_{j=0}^{i}\frac{(xy/z)^{2j}\binom{2i}{2j}\Gamma(j+1/2)}{\Gamma(j+\nu+1)}\right]$$
$$=\frac{(-z^2)^{m-1}(xy/2)^{\nu}}{(2m-2)!\sqrt{\pi}}\sum_{k=0}^{m-1}\frac{(xy/z)^{2k}\binom{2m-2}{2k}(\psi(2m-2k-1)+\gamma-\log z)\Gamma(k+1/2)}{\Gamma(\nu+k+1)}$$
$$+\frac{(xy/2)^{\nu}}{\sqrt{\pi}}\sum_{k=0}^{m-2}\frac{(-z^2)^k\zeta(2m-2k-1)}{(2k)!}\sum_{j=0}^{k}\frac{(xy/z)^{2j}\binom{2k}{2j}\Gamma(j+1/2)}{\Gamma(j+\nu+1)},$$

and

$$R_{2m-1,\nu}(x,y,z)=\lim_{\sigma\to 2m-1}\frac{\sqrt{\pi}(xy/2)^{\nu}z^{\sigma-1}}{2\Gamma(\sigma)\cos(\pi\sigma/2)}\sum_{j=m}^{\infty}\binom{\sigma-1}{2j}\frac{(xy/z)^{2j}\Gamma(j+1/2)}{\Gamma(j+\nu+1)}$$
$$=-\frac{(-z^2)^{m-1}(xy/2)^{\nu}}{(2m-1)!\sqrt{\pi}}\sum_{j=m}^{\infty}\frac{(xy/z)^{2j}\Gamma(j+1/2)}{\binom{2j}{2m-1}\Gamma(\nu+j+1)}.$$

Finally, we obtain

$$S^{J,\cos}_{2m-1+\nu}=\sum_{n=1}^{\infty}\frac{J_{\nu}(nxy)}{n^{2m-1+\nu}}\cos nz=\Phi_{2m-1,\nu}(x,y,z)+R_{2m-1,\nu}(x,y,z)$$
$$+\frac{(xy/2)^{\nu}}{\sqrt{\pi}}\sum_{i=m}^{\infty}\frac{(-z^2)^i\zeta(2m-1-2i)}{(2i)!}\sum_{j=0}^{i}\frac{(xy/z)^{2j}\binom{2i}{2j}\Gamma(j+1/2)}{\Gamma(j+\nu+1)},$$

which holds for $(x,z)\in K_1$ (see Table 2 on the page 4).

Example 1 Let $m=3$. Then we have

$$\sum_{n=1}^{\infty}\frac{J_{\nu}(nxy)}{n^{7+\nu}}\cos nz=\frac{(xy/2)^{\nu}}{\sqrt{\pi}}\left(-\frac{49\,z^6}{14400\,\Gamma(\nu+1)}-\frac{25\,(xy)^2\,z^4}{1152\,\Gamma(\nu+2)}-\frac{3\,(xy)^4\,z^2}{128\,\Gamma(\nu+3)}\right.$$
$$+\left(\frac{z^6}{720\,\Gamma(\nu+1)}+\frac{(xy)^2\,z^4}{96\,\Gamma(\nu+2)}+\frac{(xy)^4\,z^2}{64\,\Gamma(\nu+3)}+\frac{(xy)^6}{384\,\Gamma(\nu+4)}\right)\log z$$
$$+\left(\frac{z^4}{24\,\Gamma(\nu+1)}+\frac{(xy)^2\,z^2}{8\,\Gamma(\nu+2)}+\frac{(xy)^4}{32\,\Gamma(\nu+3)}\right)\zeta(3)-\left(\frac{z^2}{2\,\Gamma(\nu+1)}\right.$$
$$\left.+\frac{(xy)^2}{4\,\Gamma(\nu+2)}\right)\zeta(5)+\frac{\zeta(7)}{\Gamma(\nu+1)}+\frac{z^6}{7!}\sum_{j=4}^{\infty}\frac{(xy/z)^{2j}\Gamma(j+1/2)}{\binom{2j}{7}\Gamma(\nu+j+1)}$$
$$\left.+\sum_{i=4}^{\infty}\frac{(-1)^i\zeta(7-2i)}{(2i)!}\sum_{j=0}^{i}\frac{(xy)^{2j}z^{2i-2j}\binom{2i}{2j}\Gamma(j+1/2)}{\Gamma(j+\nu+1)}\right).$$

3.2. *Closed form cases*

The summation formula (16) takes a closed form in certain cases. Namely, the series on the right-hand side truncates because of the vanishing of F functions, i.e. $\alpha-\nu-d=2m$ if $F=\zeta,\eta,\lambda$ and $\alpha-\nu-d=2m-1$ if $F=\beta$ ($m\in\mathbb{N}$), so that we write $\alpha=\nu+2m+d-\varepsilon$, and obtain

$$\sum_{n=1}^{\infty}\frac{(s)^{n-1}\varphi_\nu((an-b)xy)f((an-b)z)}{(an-b)^{\nu+2m+d-\varepsilon}}$$
$$=\frac{(-1)^{\delta(\delta-d)}c\sqrt{\pi}(xy/2)^\nu}{2\Gamma(2m+d-\varepsilon)h(m\pi+\pi(d-\varepsilon)/2)}$$
$$\times\sum_{j=0}^{m}\binom{2m+d-\varepsilon-1}{2j+\delta}z^{2m+d-\varepsilon-2j-\delta-1}(xy)^{2j+\delta}G_j$$
$$+\frac{(xy/2)^\nu}{\sqrt{\pi}}\sum_{i=0}^{m}\frac{(-1)^{\delta(\delta-d)+i}F(2m-2i-\varepsilon)}{(2i+d)!}$$
$$\times\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}(xy)^{2j+\delta}G_j, \tag{17}$$

where $\varphi_\nu=\left\{\begin{smallmatrix}J_\nu\\ \mathbf{H}_\nu\end{smallmatrix}\right\}$ $g=\left\{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}0\\ 1\end{smallmatrix}\right\}$, $d=\left\{\begin{smallmatrix}0 & f=g\\ 1 & f\neq g\end{smallmatrix}\right.$, $h=\left\{\begin{smallmatrix}\cos & f=g\\ \sin & f\neq g\end{smallmatrix}\right.$, $\varepsilon=0$ if $F=\zeta,\eta,\lambda$ and $\varepsilon=1$ if $F=\beta$. The parameters a,b,s,c,F and convergence regions are read from Table 2. Particular closed form cases found in the literature (see Example 6), can be obtained from Equation (17).

Example 2 Consider the formula (74.1.19) in [2]

$$\sum_{n=1}^{\infty}\frac{J_\nu(nx)}{n^{\nu+2}}\cos nx=\frac{1}{3\Gamma(\nu+2)}2^{-\nu-3}x^\nu[3x^2+(\nu+1)(6x^2-12\pi x+4\pi^2)],$$

where $0<x<\pi$, $\Re\nu>-3/2$. The same result can be obtained by means of Equation (17) for $z=x$, $y=1$, $\Re\nu>-1/2$, taking $\varphi_\nu=J_\nu$, $a=1$, $b=0$, $s=1$, $c=1$, $F=\zeta$, $\varepsilon=0$, $\delta=0$, $d=0$, $f=\cos$, $h=\cos$, $m=1$, $\alpha=\nu+2$. Similarly, the sums (74.1.20), (74.1.21) and (74.1.22) from [2] are contained in Equation (17).

Example 3 By means of the formula (17) we can find the sums for the series that are not known in the literature. If we consider only finite sums, there are no results for the series with $\alpha-\nu\geq 3$, $\alpha-\nu\in\mathbb{N}_0$, whereas Equation (17) contains these cases too. For example, for the choice of parameters $\alpha-\nu=3$, $\varphi=J$, $f=\sin$, $a=2$, $b=1$, $s=1$, from the formula (17) one obtains

$$\sum_{n=1}^{\infty}\frac{J_\nu((2n-1)x)}{(2n-1)^{\nu+3}}\sin(2n-1)z=\frac{(\pi-z)\pi}{2^{\nu+3}\Gamma(\nu+1)}x^\nu z-\frac{\pi}{2^{\nu+4}\Gamma(\nu+2)}x^{\nu+2},$$

for $\nu>-1/2$, where the convergence region is K_3.

Example 4 If we now choose $\alpha-\nu=4$, $\varphi=J$, $f=\cos$, $a=1$, $b=0$, $s=-1$, Equation (17) becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}J_\nu(nx)}{n^{\nu+4}}\cos nz=\frac{(7\pi^4-30z^2\pi^2+15z^4)x^\nu}{45\cdot 2^{\nu+4}\Gamma(\nu+1)}+\frac{(3z^2-\pi^2)x^{\nu+2}}{3\cdot 2^{\nu+4}\Gamma(\nu+2)}+\frac{x^{\nu+4}}{2^{\nu+6}\Gamma(\nu+3)},$$

for $\nu>-1/2$. The convergence region is K_2.

Example 5 If we finally choose parameters: $\alpha=5, \nu=2, \varphi=\mathbf{H}, f=\sin, \ a=2, \ b=1, s=-1$, we obtain a sum

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\mathbf{H}_2((2n-1)x)}{(2n-1)^5}\sin(2n-1)z=\frac{1}{30}x^3z,$$

valid in the convergence region K_4. Note that this result coming out of the formula (17) cannot be found in the literature for $\alpha-\nu=3$.

4. Sum of the series (1)

Now, we make use of Equation (16), which is, as we have shown, the formula for finding sum of the series (5), i.e. the right-hand series of Equation (4). Thus we obtain the summation formula of the left-hand side series in Equation (4), which is actually the summation formula for the series (1):

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{(s)^{n-1}D_\nu((an-b)x)}{(an-b)^\alpha}f((an-b)z)\\ &=\frac{(-1)^{\delta(\delta-d)}c\sqrt{\pi}(x/2)^\nu}{2\Gamma(\alpha-\nu)h(\pi(\alpha-\nu)/2)}\sum_{j=0}^{\infty}\binom{\alpha-\nu-1}{2j+\delta}z^{\alpha-\nu-2j-1-\delta}x^{2j+\delta}I_{\nu+2j+\delta}\\ &\quad+\frac{(x/2)^\nu}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{(-1)^{\delta(\delta-d)+i}F(\alpha-\nu-2i-d)}{(2i+d)!}\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}x^{2j+\delta}I_{\nu+2j+\delta},\end{aligned} \tag{18}$$

where $D_\nu=\left\{\begin{smallmatrix}B_{\nu,\phi}\\ S_{\nu,\phi}\end{smallmatrix}\right\}$ $g=\left\{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}0\\ 1\end{smallmatrix}\right\}$, $d=\left\{\begin{smallmatrix}0 & f=g\\ 1 & f\neq g\end{smallmatrix}\right.$ and $h=\left\{\begin{smallmatrix}\cos & f=g\\ \sin & f\neq g\end{smallmatrix}\right.$, and for the sake of simplicity and brevity, we have denoted $I_{\nu+2j+\delta}=G_j\int_0^1\phi(y)y^{\nu+2j+\delta}\,\mathrm{d}y$. The parameters a, b, s, c, F as well as convergence regions (which we determined earlier) are read from Table 2.

4.1. *Limiting value cases*

Very important particular cases of the formula (18) ensue if $h=\sin$ and $\alpha-\nu=2m$ or $h=\cos$ and $\alpha-\nu=2m-1$, $m\in\mathbb{N}$, when the first term of Equation (18) has zero as a divisor, so we have to deal with a limiting value. We denote $\sigma=\alpha-\nu$ and replace α with $\sigma+\nu$ in Equation (18). Afterwards, choosing, for instance, $a=2$, $b=1$, $s=1$, $c=1/2$, $F=\lambda$, $\varphi_\nu=\mathbf{H}_\nu$, $g=\sin$, $\delta=1$, $f=\cos$, $d=1$, $h=\sin$, and finally $\phi(y)=y^{-1}$, we have

$$\begin{aligned}I_{\sigma+\nu}^{S_\phi,\cos}&=\frac{\sqrt{\pi}(x/2)^\nu}{4\Gamma(\sigma)\sin(\pi\sigma/2)}\sum_{j=0}^{\infty}\binom{\sigma-1}{2j+1}\frac{x^{2j+1}z^{\sigma-2j-2}j!}{(\nu+2j+1)\Gamma(j+\nu+3/2)}\\ &\quad+\frac{(x/2)^\nu}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{(-1)^i\lambda(\sigma-2i-1)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j+1}\frac{x^{2j+1}z^{2i-2j}j!}{(\nu+2j+1)\Gamma(j+\nu+3/2)},\end{aligned}$$

so that we find

$$\Phi_{2m,\nu}(x,z) = \lim_{\sigma\to 2m}\left[\frac{\sqrt{\pi}(x/2)^\nu}{4\Gamma(\sigma)\sin(\pi\sigma/2)}\sum_{j=0}^{m-1}\binom{\sigma-1}{2j+1}\frac{x^{2j+1}z^{\sigma-2j-2}j!}{(\nu+2j+1)\Gamma(j+\nu+3/2)}\right.$$
$$\left.+\frac{(x/2)^\nu}{\sqrt{\pi}}\sum_{i=0}^{m-1}\frac{(-1)^i\lambda(\sigma-2i-1)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j+1}\frac{x^{2j+1}z^{2i-2j}j!}{(\nu+2j+1)\Gamma(j+\nu+3/2)}\right]$$
$$=\frac{(x/2)^\nu}{\sqrt{\pi}}\left[\frac{(-1)^m}{2}\sum_{i=0}^{m-1}\frac{(\log(z/2)-\psi(2m-2i-1)-\gamma)x^{2i+1}z^{2m-2i-2}i!}{(\nu+2i+1)(2m-2i-2)!(2i+1)!\Gamma(\nu+i+3/2)}\right.$$
$$\left.+\sum_{i=0}^{m-2}\frac{(-1)^i\lambda(2m-2i-1)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j+1}\frac{x^{2j+1}z^{2i-2j}j!}{(\nu+2j+1)\Gamma(\nu+j+3/2)}\right],$$

and

$$R_{2m,\nu}(x,z) = \lim_{\sigma\to 2m}\frac{\sqrt{\pi}(x/2)^\nu}{4\Gamma(\sigma)\sin(\pi\sigma/2)}\sum_{j=m}^{\infty}\binom{\sigma-1}{2j+1}\frac{x^{2j+1}z^{\sigma-2j-2}j!}{(\nu+2j+1)\Gamma(j+\nu+3/2)}$$
$$=\frac{(-1)^m(x/2)^\nu}{2(2m)!\sqrt{\pi}}\sum_{j=m}^{\infty}\frac{x^{2j+1}z^{2m-2j-2}j!}{\binom{2j+1}{2m}(\nu+2j+1)\Gamma(\nu+j+3/2)}.$$

Finally, the series involving the product of a Struve integral and cosine is as follows:

$$I_{2m+\nu}^{S_\phi,\cos} = \sum_{n=1}^{\infty}\frac{\cos(2n-1)z}{(2n-1)^{2m+\nu}}\int_0^1\frac{\mathbf{H}_\nu((2n-1)xy)}{y}\,\mathrm{d}y = \Phi_{2m,\nu}(x,z)+R_{2m,\nu}(x,z)$$
$$+\frac{(x/2)^\nu}{\sqrt{\pi}}\sum_{i=m}^{\infty}\frac{(-1)^i\lambda(2m-2i-1)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j+1}\frac{x^{2j+1}z^{2i-2j}j!}{(\nu+2j+1)\Gamma(\nu+j+3/2)}$$

where $(x,z)\in K_3$ (see Table 2 on the page 4).

Example 6 If we choose $m=2$, then we obtain

$$\sum_{n=1}^{\infty}\frac{\cos(2n-1)z}{(2n-1)^{4+\nu}}\int_0^1\frac{\mathbf{H}_\nu((2n-1)xy)}{y}\,\mathrm{d}y$$
$$=\frac{(x/2)^\nu}{\sqrt{\pi}}\left[\frac{x((\log(z/2)-(3/2))z^2+(7/2)\zeta(3))}{4(\nu+1)\Gamma(\nu+3/2)}\right.$$
$$+\frac{1}{48}\sum_{j=2}^{\infty}\frac{x^{2j+1}z^{2-2j}j!}{\binom{2j+1}{4}(\nu+2j+1)\Gamma(\nu+j+3/2)}+\frac{x^3\log(z/2)}{12(\nu+3)\Gamma(\nu+5/2)}$$
$$\left.+\sum_{i=2}^{\infty}\frac{(-1)^i\lambda(3-2i)}{(2i+1)!}\sum_{k=0}^{i}\binom{2i+1}{2k+1}\frac{x^{2k+1}z^{2i-2k}k!}{(\nu+2k+1)\Gamma(k+\nu+3/2)}\right].$$

Similarly, we can obtain the other particular cases, without having previously to calculate the integral involved.

4.2. *Closed form cases*

The infinite series (18) is brought in closed form if $F=\zeta,\eta,\lambda$ and $\alpha-\nu-d=2m$ or $F=\beta$ and $\alpha-\nu-d=2m-1$ ($m\in\mathbb{N}$), so that we write $\alpha=\nu+2m+d-\varepsilon$, and have

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{(s)^{n-1}D_\nu((an-b)x)}{(an-b)^{\nu+2m+d-\varepsilon}}f((an-b)z)\\&=\frac{(-1)^{\delta(\delta-d)}c\sqrt{\pi}(x/2)^\nu}{2\Gamma(2m+d-\varepsilon)h(m\pi+\pi(d-\varepsilon)/2)}\\&\quad\times\sum_{j=0}^{m}\binom{2m+d-\varepsilon-1}{2j+\delta}z^{2m+d-\varepsilon-2j-1-\delta}x^{2j+\delta}I_{\nu+2j+\delta}\\&\quad+\frac{(x/2)^\nu}{\sqrt{\pi}}\sum_{i=0}^{m}\frac{(-1)^{\delta(\delta-d)+i}F(2m-2i-\varepsilon)}{(2i+d)!}\\&\quad\times\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}x^{2j+\delta}I_{\nu+2j+\delta},\end{aligned}\tag{19}$$

where $D_\nu=\left\{\begin{smallmatrix}B_{\nu,\phi}\\S_{\nu,\phi}\end{smallmatrix}\right\}$ $g=\left\{\begin{smallmatrix}\cos\\\sin\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}$, $d=\begin{cases}0 & f=g\\1 & f\neq g\end{cases}$ and $h=\begin{cases}\cos & f=g\\\sin & f\neq g\end{cases}$, $\varepsilon=0$ if $F=\zeta,\eta,\lambda$ and $\varepsilon=1$ if $F=\beta$. The parameters a,b,s,c,F and convergence regions are read from Table 2.

Example 7 First we take $a=1$, $b=0$, $s=1$ in Equation (19), there follows $c=1$ and $F=\zeta$ (see Table 2), then $\varepsilon=0$. For $D_2=S_{2,\phi}$, there must be $g=\sin$, which means $\delta=1$. If we take $f=\cos$, then $h=\sin$ and $d=1$ because $f\neq g$. We choose $\phi(y)=\operatorname{ctg}y$. Let $m=1$. The function $\operatorname{ctg}y$ is unbounded in the neighbourhood of 0 and the integral $\int_0^1\operatorname{ctg}y\,\mathrm{d}y$ does not converge, so we cannot apply Lemma 1. However, for $0<|x|<\pi$, $\lim_{y\to0+}\operatorname{ctg}y\mathbf{H}_2(nxy)=0$, where $n\in\mathbb{N}$, the function $\operatorname{ctg}y\mathbf{H}_2(nxy)$ is integrable with respect to $y\in(0,1)$, and we have

$$\begin{aligned}\left|x\int_0^1\operatorname{ctg}y\mathbf{H}_2(nxy)\,\mathrm{d}y\right|&=\left|\int_0^x\operatorname{ctg}\frac{t}{x}\mathbf{H}_2(nt)\,\mathrm{d}t\right|\leqslant\int_0^x\left|\operatorname{ctg}\frac{t}{x}\mathbf{H}_2(nt)\right|\,\mathrm{d}t\\&\leqslant\int_0^\pi\left|\operatorname{ctg}\frac{t}{\pi}\mathbf{H}_2(nt)\right|\,\mathrm{d}t.\end{aligned}$$

Also, $|\operatorname{tg}(t/\pi)|>|t|/\pi$ implies $|\operatorname{ctg}(t/\pi)|<\pi/|t|$, for $|t|<\pi$. Additionally, we find

$$|\mathbf{H}_2(nt)|=\frac{2n|t|}{3\pi}\left|1-\frac{3\pi J_1(nt)}{2nt}+\frac{3\pi J_2(nt)}{n^2t^2}\right|<\frac{2n|t|}{3\pi}\left(1+\frac{3\pi}{8}\right)$$

So,

$$\left|\operatorname{ctg}\frac{t}{\pi}\mathbf{H}_2(nt)\right|=\left|\operatorname{ctg}\frac{t}{\pi}\right|\cdot|\mathbf{H}_2(nt)|<\frac{\pi}{|t|}\cdot\frac{2n|t|}{3\pi}\left(1+\frac{3\pi}{8}\right)=\frac{2n}{3}\left(1+\frac{3\pi}{8}\right),$$

and there follows

$$\int_0^\pi\left|\operatorname{ctg}\frac{t}{\pi}\mathbf{H}_2(nt)\right|\,\mathrm{d}t<\frac{2n}{3}\left(1+\frac{3\pi}{8}\right)\int_0^\pi\,\mathrm{d}t=\frac{2n\pi}{3}\left(1+\frac{3\pi}{8}\right),$$

which means that $\left|\int_0^1\operatorname{ctg}y\mathbf{H}_2(nxy)\,\mathrm{d}y\right|<n\pi(8+3\pi)/12|x|=M_n(x)$, and we can easily see that for each x, $0<|x|<\pi$, the sequence $M_n(x)/n^3\to0$ monotonically, so that Lemma 3 may be applied. Making use of Equation (13), we obtain

$$\sum_{n=1}^{\infty}\frac{\cos nz}{n^5}\int_0^1 \operatorname{ctg} y\mathbf{H}_2(nxy)\,\mathrm{d}y$$
$$=\frac{x^3}{1156\sqrt{\pi}}((8\pi^2-24\pi z+12z^2)\log(2\sin 1)+(12\pi^2-36\pi z+18z^2)(\mathrm{Cl}_2(2)+\mathrm{Cl}_3(2))$$
$$-(6\pi^2-18\pi z+9z^2)\mathrm{Cl}_4(2))+\frac{x^5}{4608\sqrt{\pi}}\big(\log(2\sin 1)+10\mathrm{Cl}_2(2)+20\mathrm{Cl}_3(2)$$
$$-30(\mathrm{Cl}_4(2)+\mathrm{Cl}_5(2))+15\mathrm{Cl}_6(2)\big),$$

where $|x|<\pi$ and $|x|<z<2\pi-|x|$ (see Table 2). On the right-hand side are Clausen functions defined by [1]

$$\mathrm{Cl}_{2\nu}(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2\nu}},\quad \mathrm{Cl}_{2\nu-1}(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2\nu-1}},\quad \nu\in\mathbb{N}.$$

Example 8 Let $a=1$, $b=0$, $s=-1$ in Equation (19), implying $c=0$ and $F=\eta,\varepsilon=0$. For $D_{1/2}=B_{1/2,\phi}$ there must be $g=\cos$ and $\delta=0$. Further, we take $f=\cos$ implying $h=\cos$, $d=0$ because $f=g$. Let $m=2$. We choose $\phi(y)=(1-y^2)^{-1/2}$. It is unbounded about 1, but integrable on $(0,1)$, thus satisfying conditions of Lemma 2. So applying Equation (19), we find

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nz}{n^{9/2}}\int_0^1\frac{J_{1/2}(nxy)}{\sqrt{1-y^2}}\,\mathrm{d}y$$
$$=\frac{\pi\sqrt{x\pi}}{4\Gamma^2(1/4)}\left(\frac{7\pi^4}{90}-\frac{\pi^2z^2}{3}+\frac{z^4}{6}-\frac{\pi^2x^2}{20}+\frac{3x^2z^2}{20}+\frac{7x^4}{720}\right),$$

where $|x|<\pi$ and $|x|-\pi<z<\pi-|x|$ (see Table 2).

We have already said that in order to find the sum of the series (1), it is not necessary to calculate the integrals (2). Besides, it does not have to be done elementarily. Yet, if we calculate the integral in Example 8, the above series takes a different form, giving rise to the following formula

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nz}{n^{9/2}}J_{1/4}^2\left(\frac{nx}{2}\right)=\frac{\sqrt{x\pi}}{2\Gamma^2(1/4)}\left(\frac{7\pi^4}{90}-\frac{\pi^2z^2}{3}+\frac{z^4}{6}-\frac{\pi^2x^2}{20}+\frac{3x^2z^2}{20}+\frac{7x^4}{720}\right),$$

whereby we obtain the sum of a new series. Generally speaking, for $\nu>-1$, there holds

$$\int_0^1\frac{J_\nu(nxy)}{\sqrt{1-y^2}}\,\mathrm{d}y=\frac{\pi}{2}J_{\nu/2}^2\left(\frac{nx}{2}\right),$$

so for this type of integrals there exists a whole class of new closed form formulas.

Example 9 Further, we take $a=2$, $b=1$, $s=1$ in Equation (19). In Table 2, we read $c=1/2$ and $F=\lambda$. So $\varepsilon=0$. If we choose $D_{1/3}=S_{1/3,\phi}$, $f=\sin$, then we have $g=\cos$, $\delta=0$, $d=1$, $h=\sin$. Let $m=1$ and $\phi(y)=\log y$, which is unbounded in the neighbourhood of 0, but integrable on $(0,1)$, so Lemma 2 holds. Applying Equation (19), we obtain

$$I_1^{S_\phi,\sin}=\sum_{n=1}^{\infty}\frac{\sin(2n-1)z}{(2n-1)^{10/3}}\int_0^1\log y\mathbf{H}_{1/3}((2n-1)xy)\,\mathrm{d}y=\frac{27\sqrt[3]{x}\,\Gamma(1/6)}{640\sqrt[3]{2}}\left(z\pi+z^2+\frac{12x^2}{275}\right),$$

where $|x|<\pi/2$ and $|x|<z<\pi-|x|$ (see Table 2).

Example 10 Finally, let in Equation (19) be $a=2,\ b=1,\ s=-1$, implying $c=0,\ F=\beta,\ \varepsilon=1$. If we choose $D_1=B_{1,\phi}$, there follows $g=\cos$ and $\delta=0$. If $f=\sin$, then $d=1$ and $h=\sin$ because $f\neq g$. Further, let $\phi(y)=y^\mu,\ \mu>-2$, and $m=1$. The integral $\int_0^1 y^\mu\,\mathrm{d}y$ does not necessarily converge for $\mu>-2$. However, making use of Equation (6), where we substitute u for $\cos t$, we come, for $|x|<\pi/2$, to the following estimate

$$y^\mu|J_1((2n-1)xy)| \leqslant \frac{2|x|}{\pi}(2n-1)y^{\mu+1}\left|\frac{\sin((2n-1)xy)}{(2n-1)xy}\right| < (2n-1)y^{\mu+1},$$

whereupon we find

$$\left|\int_0^1 J_1((2n-1)xy)y^\mu\,\mathrm{d}y\right| \leqslant \int_0^1 |J_1((2n-1)xy)|y^\mu\,\mathrm{d}y < (2n-1)\int_0^1 y^{\mu+1}\,\mathrm{d}y = \frac{2n-1}{\mu+2},$$

with $M_n(x)=(2n-1)/(\mu+2)$, so $M_n(x)/(2n-1)^3$ monotonically tends to zero when n increases to infinity, which means that Lemma 3 holds, and by applying Equation (19), we obtain

$$I_3^{B_\phi,\sin} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(2n-1)z}{(2n-1)^3}\int_0^1 J_1((2n-1)xy)\,y^\mu\mathrm{d}y = \frac{\pi xz}{8(\mu+2)},$$

$|x|<\pi/2$ and $|x|-\pi/2<z<\pi/2-|x|$ (see Table 2).

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