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LEVEL CROSSING RATE OF PRODUCT OF TWO NAKAGAMI-m RANDOM VARIABLES AND RAYLEIGH RANDOM VARIABLE

Abstract: *In this paper, the level crossing rate of product of two Nakagami random variables and Rayleigh random variable will be calculated. This result can be used in the performance analysis of wireless relay communication system with three sections: Nakagami-m fading is present in the first and second section and Rayleigh fading is present in the third section. Also, the obtained result can be used for calculation of the level crossing rate of the product of three Rayleigh random processes, and the level crossing rate of the product of two Rayleigh random processes and Nakagami-m random process. The influence of the parameter m on the level crossing rate is analysed and considered.*

Key words: Level crossing rate, Nakagami-m random variable, Rayleigh random variable.

INTRODUCTION

In this paper, the wireless relay communication system with three sections is considered. The introduction of relays allows lower transmitter power, which results in lower energy consumption and lower human exposure to the electromagnetic field. Outage probability, joint probability density function of signal envelope and its first derivatives, bit error probability, level crossing rate and average fade duration and its first derivatives are performance measures of wireless communication system [1]. Probability density function, cumulative distribution function, moment generating, the first moment, the second function and the third function are the first order performance measures of a wireless communication system. The level crossing rate, average fade duration and joint probability density function are the second order performance measures of wireless communication system. Average fade duration can be evaluated as ratio of the outage probability and the level crossing rate, and the level crossing rate can be evaluated as average value of the first derivative of random process. Level crossing rate is the number of crossings of a determined level [2]. The Rayleigh random envelope and its first derivation are independent. The first derivative of Rayleigh envelope has Gaussian distribution [3], [4]. Therefore, joint probability density function of Rayleigh envelope and its first derivative can be evaluated as the product of Rayleigh distribution and Gaussian distribution. There are more works (papers) considering and analysing products and ratios of random processes in open technical literature [5]–[11].

In this paper, the level crossing rate of product of two Nakagami-m random variables and Rayleigh random variable is evaluated. This result can be used for evaluation of average fade duration of wireless relay communication system with three sections. Nakagami-m fading is present in the first and second section and Rayleigh fading is present in the third section. The level crossing rate, calculated in this paper can be used for evaluation of level crossing rate of product of three Rayleigh random processes, and level crossing rate of product of Nakagami-m random variable and two Rayleigh random variables. To the best author's knowledge, level crossing rate of product of two Nakagami-m random variables and Rayleigh random variable is not considered in open technical literature.

PROBLEM FORMULATION

Random variables y_1 and y_2 follow Nakagami-*m* distribution [12]:

$$p_{y_1}(y_1) = \frac{2}{\Gamma(m_1)}\left(\frac{m_1}{\Omega_1}\right)^{m_1} y_1^{2m_1-1} e^{-\frac{m_1}{\Omega_1}y_1^2},\ y_1 \ge 0$$
$$p_{y_2}(y_2) = \frac{2}{\Gamma(m_2)}\left(\frac{m_2}{\Omega_2}\right)^{m_2} y_2^{2m_2-1} e^{-\frac{m_2}{\Omega_2}y_2^2},\ y_2 \ge 0, \tag{1}$$

where m_1 and m_2 are parameters higher than 0.5; Ω_1 and Ω_2 are mathematical expectations of $|y_1|^2$ and $|y_2|^2$, respectively.

Random variable y_3 is Rayleigh random variable:

$$p_{y_3}(y_3) = \frac{2y_3}{\Omega_3} e^{-\frac{y_3^2}{\Omega_3}},\ y_3 \ge 0, \tag{2}$$

where $\Omega_3 = |y_3|^2$.

In the following text, the level crossing rate of the random variable

$$y = y_1 \cdot y_2 \cdot y_3 \tag{3}$$

will be determined.

LEVEL CROSSING RATE

In order to determine the level crossing rate, the probability density function of the time derivative of *y* needs to be determined first. Therefore, the first derivative of y is equal to

$$\dot{y} = \dot{y}_1 y_2 y_3 + y_1 \dot{y}_2 y_3 + y_1 y_2 \dot{y}_3 \quad (4)$$

Having in mind that the first derivative of Nakagami-m or Rayleigh random variable is a zero-mean Gaussian random variable [13], and that linear transformation of Gaussian random variables is also a Gaussian random variable, the first derivative $\dot{y}$ is Gaussian random variable with mean and variance to be determined.
The mean of $\dot{y}$ is:

$$\overline{\dot{y}} = \overline{\dot{y}}_1 y_2 y_3 + y_1 \overline{\dot{y}}_2 y_3 + y_1 y_2 \overline{\dot{y}}_3 \quad (5)$$

Since $\dot{y}_1, \dot{y}_2, \dot{y}_3$ are Gaussian variables with zero mean, i.e. $\overline{\dot{y}}_1, = \overline{\dot{y}}_2 = \overline{\dot{y}}_3 = 0$, the mean of $\dot{y}$ is $\overline{\dot{y}} = 0$.
The variance of $\dot{y}$ is:

$$\sigma_{\dot{y}}^2 = y_2^2 y_3^2 \sigma_{\dot{y}_1}^2 + y_1^2 y_3^2 \sigma_{\dot{y}_2}^2 + y_1^2 y_2^2 \sigma_{\dot{y}_3}^2 \quad (6)$$

where [13]

$$\sigma_{\dot{y}_1}^2 = \pi^2 f_m^2 \frac{\Omega_1}{m_1},\ \sigma_{\dot{y}_2}^2 = \pi^2 f_m^2 \frac{\Omega_2}{m_2},\ \sigma_{\dot{y}_3}^2 = \pi^2 f_m^2 \Omega_3, \quad (7)$$

and f_m is the maximum Doppler frequency.
After substituting Eqn. (7) in Eqn. (6), the expression for variance becomes:

$$\begin{aligned}\sigma_{\dot{y}}^2 &= \pi^2 f_m^2 \left(y_2^2 y_3^2 \frac{\Omega_1}{m_1} + y_1^2 y_3^2 \frac{\Omega_2}{m_2} + y_1^2 y_2^2 \Omega_3 \right) \\ &= \pi^2 f_m^2 y_2^2 y_3^2 \frac{\Omega_1}{m_1} \left(1 + \frac{y_1^2}{y_2^2} \frac{\Omega_2}{m_2} \frac{m_1}{\Omega_1} + \frac{y_1^2}{y_3^2} \Omega_3 \frac{m_1}{\Omega_1} \right).\end{aligned} \quad (8)$$

From Eqn. (3) we have

$$y_1 = \frac{y}{y_2 y_3} \ . \quad (9)$$

Equation (8), with the help of Eqn. (9), becomes

$$\sigma_{\dot{y}}^2 = \pi^2 f_m^2 y_2^2 y_3^2 \frac{\Omega_1}{m_1} \left(1 + \frac{y^2}{y_2^4 y_3^2} \frac{\Omega_2}{m_2} \frac{m_1}{\Omega_1} + \frac{y^2}{y_2^2 y_3^4} \Omega_3 \frac{m_1}{\Omega_1} \right). \quad (10)$$

Joint probability density function of $y, \dot{y}, y_2, y_3$ is:

$$p_{y\dot{y}y_2y_3}\left(y, \dot{y}, y_2, y_3\right) = p_{\dot{y}}\left(\dot{y} \big/ y y_2 y_3\right) p_{yy_2y_3}\left(y, y_2, y_3\right), \quad (11)$$

where

$$p_{yy_2y_3}\left(y, y_2, y_3\right) = p_y\left(y \big/ y_2 y_3\right) p_{y_2}\left(y_2\right) p_{y_3}\left(y_3\right) \quad (12)$$

Conditional probability density function from Eqn. (12) is equal to [14]

$$\begin{aligned}p_y\left(y \big/ y_2 y_3\right) &= \frac{1}{\frac{dy}{dy_1}} p_{y_1}\left(\frac{y}{y_2 y_3}\right) \\ &= \frac{1}{y_2 y_3} p_{y_1}\left(\frac{y}{y_2 y_3}\right)\end{aligned} \quad (13)$$

After substituting Eqns. (13) and (12) in Eqn. (11), the expression for the joint probability density function of y and $\dot{y}$ may be written as

$$\begin{aligned}p_{y\dot{y}}\left(y, \dot{y}\right) &= \int_0^\infty dy_2 \int_0^\infty dy_3 \cdot p_{y\dot{y}y_2y_3}\left(y, \dot{y}, y_2, y_3\right) \\ &= \int_0^\infty dy_2 \int_0^\infty dy_3 p_{\dot{y}}\left(\dot{y} \big/ y y_2 y_3\right) \cdot \frac{1}{y_2 y_3} \\ &\quad \cdot p_{y_1}\left(\frac{y}{y_2 y_3}\right) p_{y_2}\left(y_2\right) p_{y_3}\left(y_3\right).\end{aligned} \quad (14)$$

Finally, the level crossing rate may be expressed as

$$\begin{aligned}N_y(y) &= \int_0^\infty \dot{y} \cdot p_{y\dot{y}}(y, \dot{y}) d\dot{y} \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \dot{y} p_{\dot{y}}\left(\dot{y} \big/ y y_2 y_3\right) \cdot \frac{1}{y_2 y_3} \cdot p_{y_1}\left(\frac{y}{y_2 y_3}\right) \\ &\quad \cdot p_{y_2}\left(y_2\right) p_{y_3}\left(y_3\right) dy_2 dy_3 d\dot{y} \\ &= \int_0^\infty \int_0^\infty \frac{1}{y_2 y_3} \cdot p_{y_1}\left(\frac{y}{y_2 y_3}\right) \cdot p_{y_2}\left(y_2\right) p_{y_3}\left(y_3\right) \\ &\quad \cdot \int_0^\infty \dot{y} p_{\dot{y}}\left(\dot{y} \big/ y y_2 y_3\right) d\dot{y} dy_2 dy_3\end{aligned} \quad (15)$$

Bearing in mind that

$$\int_0^\infty \dot{y} p_{\dot{y}}\left(\dot{y} \big/ y y_2 y_3\right) d\dot{y} = \frac{\sigma_{\dot{y}}}{\sqrt{2\pi}}, \quad (16)$$

the expression for the level crossing rate becomes

$$\begin{aligned}N_y &= \int_0^\infty dy_2 \int_0^\infty dy_3 \cdot \frac{\sigma_{\dot{y}}}{\sqrt{2\pi}} \frac{1}{y_2 y_3} p_{y_1}\left(\frac{y}{y_2 y_3}\right) \\ &\quad \cdot p_{y_2}\left(y_2\right) p_{y_3}\left(y_3\right).\end{aligned} \quad (17)$$

The final expression for the level crossing rate may be obtained by inserting Eqns. (1), (2) and (10) into Eqn. (17):

$$\begin{aligned}N_y &= \int_0^\infty dy_2 \int_0^\infty dy_3 \cdot \frac{1}{\sqrt{2\pi}} \cdot \pi f_m \cdot y_2 y_3 \frac{\Omega_1^{1/2}}{m_1^{1/2}} \\ &\quad \cdot \left(1 + \frac{y^2}{y_2^4 y_3^2} \frac{\Omega_2}{m_2} \frac{m_1}{\Omega_1} + \frac{y^2}{y_2^2 y_3^4} \Omega_3 \frac{m_1}{\Omega_1} \right)^{1/2} \cdot \\ &\quad \cdot \frac{1}{y_2 y_3} \cdot \frac{2}{\Gamma(m_1)} \left(\frac{m_1}{\Omega_1} \right)^{m_1} \cdot \frac{y^{2m_1 - 1}}{y_2^{2m_1 - 1} y_3^{2m_1 - 1}} \cdot e^{-\frac{m_1}{\Omega_1} \cdot \frac{y^2}{y_2^2 y_3^2}} \\ &\quad \cdot \frac{2}{\Gamma(m_2)} \left(\frac{m_2}{\Omega_2} \right)^{m_2} \cdot y_2^{2m_2 - 1} \cdot e^{-\frac{m_2}{\Omega_2} \cdot y^2} \cdot \frac{2 y_3}{\Omega_3} \cdot e^{-\frac{y_3^2}{\Omega_3}} \\ &= \frac{1}{\sqrt{2\pi}} \pi f_m \frac{2}{\Gamma(m_1)} \left(\frac{m_1}{\Omega_1} \right)^{m_1} \frac{2}{\Gamma(m_2)} \left(\frac{m_2}{\Omega_2} \right)^{m_2} \\ &\quad \cdot \frac{\Omega_1^{1/2}}{m_1^{1/2}} \frac{2}{\Omega_3} y^{2m_1 - 1}\end{aligned}$$

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$$\int_0^\infty dy_2 \int_0^\infty dy_3 \cdot \left(1+\frac{y^2}{y_2^4 y_3^2}\frac{\Omega_2}{m_2}\frac{m_1}{\Omega_1}+\frac{y^2}{y_2^2 y_3^4}\Omega_3\frac{m_1}{\Omega_1}\right)^{1/2}$$
$$\cdot \exp\left(-\frac{m_1}{\Omega_1}\frac{y^2}{y_2^2 y_3^2}-\frac{m_2}{\Omega_2}y_2^2-\frac{y_3^2}{\Omega_3}\right. \tag{18}$$
$$\left.+2(m_2-m_1)\ln y_2-2(m_1-1)\ln y_3\right).$$

Two-fold integral in Eqn. (18) may be solved by using Laplace approximation theorem [15] for two-fold integrals:

$$\int_0^\infty dy_2 \int_0^\infty dy_3 g\left(y_2, y_3\right) e^{-\lambda \cdot f\left(y_2, y_3\right)} = \frac{\pi}{\lambda} e^{-\lambda \cdot f\left(y_{20}, y_{30}\right)} \cdot \frac{g\left(y_{20}, y_{30}\right)}{\sqrt{B\left(y_{20}, y_{30}\right)}}, \tag{19}$$

where y_{20} and y_{30} are solutions of the following set of equations:

$$\frac{\partial f\left(y_{20}, y_{30}\right)}{\partial y_{20}}=0, \quad \frac{\partial f\left(y_{20}, y_{30}\right)}{\partial y_{30}}=0, \tag{20}$$

and

$$B\left(y_{20}, y_{30}\right)=\begin{bmatrix} \frac{\partial^2 f\left(y_{20}, y_{30}\right)}{\partial y_{20}^2} & \frac{\partial^2 f\left(y_{20}, y_{30}\right)}{\partial y_{20}\partial y_{30}} \\ \frac{\partial^2 f\left(y_{20}, y_{30}\right)}{\partial y_{20}\partial y_{30}} & \frac{\partial^2 f\left(y_{20}, y_{30}\right)}{\partial y_{30}^2} \end{bmatrix}. \tag{21}$$

Constant $\lambda = 1$, and functions f and g are equal to

$$f(y_2, y_3)=\left(\frac{m_1}{\Omega_1}\frac{y^2}{y_2^2 y_3^2}+\frac{m_2}{\Omega_2}y_2^2+\frac{y_3^2}{\Omega_3}\right. \tag{22}$$
$$\left.-2(m_2-m_1)\ln y_2+2(m_1-1)\ln y_3\right),$$

$$g(y_2, y_3)=\left(1+\frac{y^2}{y_2^4 y_3^2}\frac{\Omega_2}{m_2}\frac{m_1}{\Omega_1}+\frac{y^2}{y_2^2 y_3^4}\Omega_3\frac{m_1}{\Omega_1}\right)^{1/2}. \tag{23}$$

NUMERICAL RESULTS

Numerical results illustrate the influence of different parameters on the level crossing rate. The level crossing rate is normalized with the maximum Doppler frequency. In Figures 1, 2 and 3, the level crossing rate of the product of two Nakagami-m random processes and Rayleigh random process is shown. The level crossing rate increases for lower values of resulting signal envelope and the level crossing rate decreases for higher values of resulting signal envelope. The influence of signal envelope on level crossing rate is higher for lower values of signal envelope. The influence of Nakagami-*m* parameter on level crossing rate is higher for lower values of parameter *m*. Also, the influence of parameter *m* on the level crossing rate is higher for lower values of resulting signal envelope.

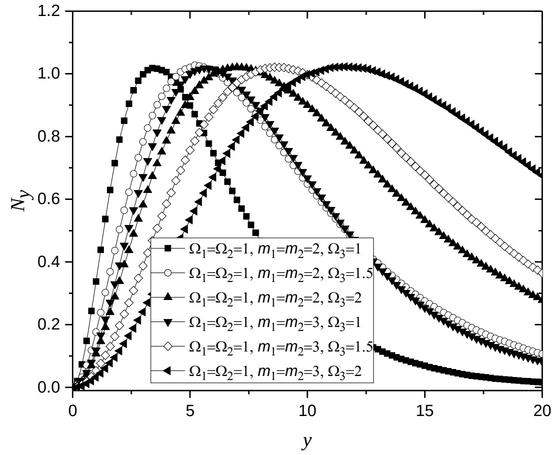

Figure 1. *The level crossing rate of product of two Nakagami-m random processes and Rayleigh random process for different m_1 and m_2*

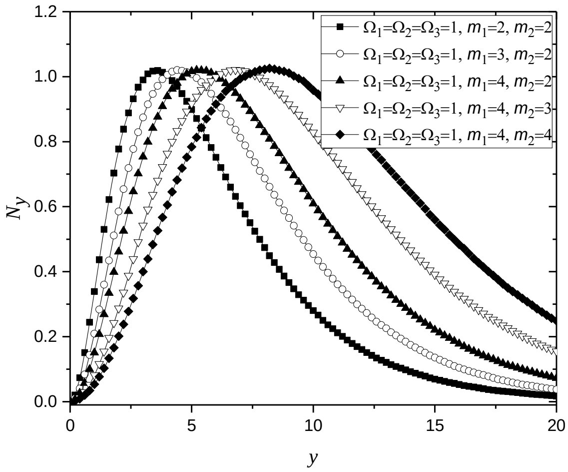

Figure 2. *The level crossing rate of product of two Nakagami-m random processes and Rayleigh random process for different Ω_3*

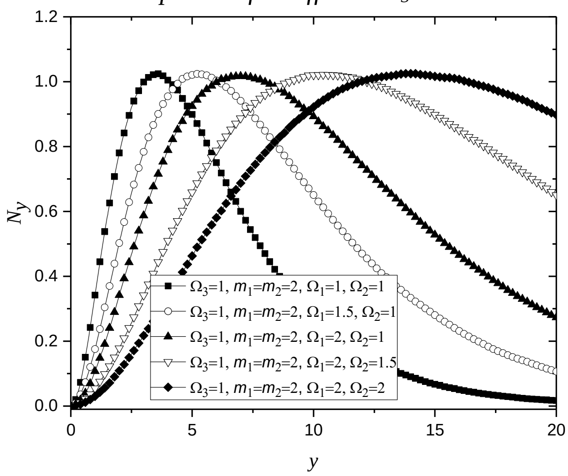

Figure 3. *The level crossing rate of product of two Nakagami-m random processes and Rayleigh random process for different Ω_1 and Ω_2*

The level crossing rate increases as the power of Rayleigh envelope increases. The influence of Rayleigh envelope on the level crossing rate is higher for lower values of Nakagami-*m* parameter *m*. The level crossing rate increases as the parameter *m* increases.

CONCLUSION

In this paper, the level crossing rate of the product of two Nakagami-*m* random processes and Rayleigh random process is evaluated. This result can be used for evaluation of average fade duration of wireless relay communication system with three sections, operating over multipath fading channel. Nakagami-*m* fading is present in the first and second section and Rayleigh short term fading is present in the third section. The level crossing rate obtained in this paper can be used for evaluation of the level crossing rate of product of three Rayleigh random processes, and the level crossing rate of the product of Nakagami-*m* random process and two Rayleigh random processes. The level crossing rate increases, for low values of the signal envelope, as parameter *m* increases. On the other hand, for high values of the resulting signal envelope, the level crossing rate is lower for lower values of parameter *m*.

REFERENCES

[1] J. G. Proakis: "*Digital communications"*, McGraw-Hill, 1995, New York, NY, USA.

[2] W. Jakes: "*Microwave Mobile Communications"*, 2nd ed., IEEE Press, 1993, Piscataway, NJ, USA.

[3] Z. Hadzi-Velkov, N. Zlatanov, G. K. Karagiannidis: "*Level Crossing Rate and Average Fade Duration of the Multihop Rayleigh Fading Channel*," in 2008 IEEE International Conference on Communications, 2008, pp. 4451–4455.

[4] Z. Hadzi-Velkov, N. Zlatanov, G. K. Karagiannidis: "*On the second order statistics of the multihop rayleigh fading channel*", IEEE Trans. Commun., vol. 57, no. 6, 2009, pp. 1815–1823.

[5] M. Č. Stefanović, D. Đošić, S. Maričić, M. Matović, A. Matović: "*Level crossing rate of the product of two random variables*" in INFOTEH 2013 Proceedings of papers, 2013, pp. 404–407.

[6] G. V Milovanović, M. Č. Stefanović, S. R. Panić, J. A. Anastasov, D. S. Krstić: "*Statistical analysis of the square ratio of two multivariate exponentially correlated α–μ distributions and its application in telecommunications*", Math. Comput. Model., vol. 54, no. 1, 2011, pp. 152–159.

[7] N. Zlatanov, Z. Hadzi-Velkov, G. K. Karagiannidis: "*Level crossing rate and average fade duration of the double Nakagami-m random process and application in MIMO keyhole fading channels*", IEEE Commun. Lett., vol. 12, no. 11, 2008, pp. 822–824.

[8] A. Matović, E. Mekić, N. Sekulović, M. Stefanović, M. Matović, Č. Stefanović: "*The distribution of the ratio of the products of two independent-variates and its application in the performance analysis of relaying communication systems*", Math. Probl. Eng., vol. 2013, Article ID 147106, 2013, 6 pages.

[9] E. Mekić, M. Stefanović, P. Spalević, N. Sekulović, A. Stanković: "*Statistical analysis of ratio of random variables and its application in performance analysis of multihop wireless transmissions*", Math. Probl. Eng., vol. 2012, Article ID 841092, 2012, 10 pages.

[10] A. Stanković, N. Sekulović, M. Č. Stefanović, E. Mekić, S. Zdravković: "*Statistical characteristic of ratio of product of two random Rayleigh variables and Rayleigh random variable and its application in performance analysis of wireless communication systems*", in INFOTEH 2011 Proceedings of papers, 2011, pp. 79–81.

[11] D. Č. Pavlović, N. M. Sekulović, G. V Milovanović, A. S. Panajotović, M. Č. Stefanović, Z. J. Popović: "*Statistics for Ratios of Rayleigh, Rician, Nakagami-, and Weibull Distributed Random Variables*", Math. Probl. Eng., vol. 2013, Arcticle ID 252804, 2013, 10 pages.

[12] M. Nakagami: "*The m-distribution - A General Formula of Intensity Distribution of Rapid Fading*", in Statistical Methods in Radio Wave Propagation, 1960, pp. 3–36.

[13] M. D. Yacoub, J. E. V. Bautista, L. Guerra de Rezende Guedes: "*On higher order statistics of the Nakagami-m distribution*", IEEE Trans. Veh. Technol., vol. 48, no. 3, 1999, pp. 790–794.

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Srđan Maričić was born in Belgrade, Serbia, in 1969. He is MSc in electrical engineering and he is pursuing the PhD degree in telecommunications at the University of Nis, Faculty of Electronic Engineering. His main areas of research include fading and its influence on wireless telecommunication systems. He is currently working as an assistant professor at IT Department of the Faculty of Applied Management, Economics and Finance in Belgrade.

BROJ OSNIH PRESEKA PROIZVODA DVE NAKAGAMI-M SLUČAJNE PROMENLJIVE I REJLIJEVE SLUČAJNE PROMENLJIVE

Srđan Maričić, Nenad Milošević

Rezime: *U ovom radu će biti izračunat broj osnih preseka proizvoda dve Nakagami-m slučajne promenljive i Rejlijeve slučajne promenljive. U radu je izveden približni izraz u zatvorenom obliku i ispitan uticaj parametara slučajnih promenljivih na broj osnih preseka. Ovaj rezultat se može koristiti u analizi performansi bežičnih relejnih komunikacionih sistema sa tri sekcije: Nakagami-m feding je prisutan u prvoj i drugoj sekciji, dok je Rejlijev feding prisutan u trećoj. Takođe, dobijeni rezultat se može koristiti za izračunavanje broja osnih preseka proizvoda tri slučajna Rejlijeva procesa, kao i broja osnih preseka proizvoda dve slučajna Rejlijeva procesa i Nakagami-m slučajnog procesa.*

Ključne reči: broj osnih preseka, Nakagami-m slučajna promenljiva, Rejlijeva slučajna promenljiva.